Current Progress:

So far we have focused on linear programs that are in *standard form*

$$
\begin{aligned}
\text{Maximize} \quad & 25\,X_B + 30\,X_C \\
\text{Subject to} \quad & (1/200)\,X_B + (1/140)\,X_C \le 40 \\
& 0 \le X_B \le 6000 \\
& 0 \le X_C \le 4000
\end{aligned}
$$

The original point is always a feasible cornerpoint.

The simplex method can be applied to proceeds from cornerpoint to better cornerpoint until it recognizes optimality.

In Practice:

Not all forms of LPs are just standard form LPs

$$\text{minimize } Z = 12x_1 + 5x_2 - 7x_3$$

Transformation:

$$\text{minimize } Z = 12x_1 + 5x_2 - 7x_3$$

↓ Multiply by -1

$$\text{maximize } (-Z) = -12x_1 - 5x_2 + 7x_3$$

Multiply the optimum objective function value by -1 to recover the minimum value of the original minimization objective function

Equality Constraints:

Acme Bicycle Company Problem

$$\text{maximize } Z = 15x_1 + 10x_2$$

$$\text{subject to:} \quad x_1 \leq 2$$

$$x_2 \leq 3$$

$$x_1 + x_2 = 4$$

↑

Equality Constraint

We add a dimension to the mathematical representation of the original problem

We add a non-negative artificial variable to any equality constraints

Solution:

Add slack variables for the constraints

$$
\begin{array}{lcccccccc}
(1) & x_1 & & +s_1 & & & = & 2 \\
(2) & & x_2 & & +s_2 & & = & 3 \\
(3) & x_1 & +x_2 & & & +a_1 & = & 4
\end{array}
$$

We add one slack variable for the first constraint

We add another slack variable for the second constraint

Solution:

Add slack variables for the constraints

$$
\begin{array}{lccccccc}
(1) & x_1 & & +s_1 & & & = & 2 \\
(2) & & x_2 & & +s_2 & & = & 3 \\
(3) & x_1 & +x_2 & & & \boxed{+a_1} & = & 4
\end{array}
$$

The third constraint will involve x1 and x2, and we may not be able to directly see how to set those two values such that all of the constraints are satisfied simultaneously.

We cannot add the third slack variable for the third constraint, since it is already in equality format. Therefore, we add a nonnegative ***artificial variable*** to the equality constraints.

Phase 1 LP

Phase 1: solve an LP whose objective is to minimize the value of any artificial variable in the model. For the case a1, if you can drive all of the artificial variables to zeros, then you will be at a feasible cornerpoint of the original problem.

Phase 2: starting at the feasible cornerpoint found during phase 1, switch over to the original objective function and continue iterating until the optimum point is found.

Phase 1

Solve an LP whose objective is to minimize the value of any artificial variables

$$\text{minimize } W = a_1 + a_2 + a_3 + \ldots$$

W is defined as the sum of the constraint violations

$$\text{minimize } W = a_1 + a_2 + a_3 + \ldots$$

Multiply by -1

$$\text{maximize } -W + a_1 + a_2 + a_3 + \ldots = 0$$

Initial Tableau

basic variable	eqn. no.	W	Z	x_1	x_2	s_1	s_2	a_1	RHS	MRT
W	ph1	-1	0	0	0	0	0	1	0	*never*
Z	ph2	0	1	-15	-10	0	0	0	0	*never*
s_1	1	0	0	1	0	1	0	0	2	
s_2	2	0	0	0	1	0	1	0	3	
a_1	3	0	0	1	1	0	0	1	4	

This tableau has two objective functions: W for phase 1, which seeks to minimize the sum of the artificial variables (i.e., a1)

Z for phase 2 which represents the original objective.

We need to eliminate the coefficient in the phase 1 (W) objective function row

We subtract each row containing an artificial variable from the phase 1 objective function now. We subtract equation 3 from equation ph1.

basic variable	eqn. no.	W	Z	x_1	x_2	s_1	s_2	a_1	RHS	MRT
W	ph1	-1	0	-1	-1	0	0	0	-4	*never*
Z	ph2	0	1	-15	-10	0	0	0	0	*never*
s_1	1	0	0	1	0	1	0	0	2	
s_2	2	0	0	0	1	0	1	0	3	
a_1	3	0	0	1	1	0	0	1	4	

During phase 1, we are using the phase 1 objective function, and as the tableau above shows, both x_1 and x_2 are tied for the entering basic variable with objective function coefficients of –1. Let us arbitrarily choose x_2 as the entering basic variable. The calculations are shown below.

basic variable	eqn. no.	W	Z	x_1	x_2	s_1	s_2	a_1	RHS	MRT
W	ph1	-1	0	-1	-1	0	0	0	-4	*never*
Z	ph2	0	1	-15	-10	0	0	0	0	*never*
s_1	1	0	0	1	0	1	0	0	2	no limit
s_2	2	0	0	0	1	0	1	0	3	3/1=3
a_1	3	0	0	1	1	0	0	1	4	4/1=4

The phase 1 objective function value, the sum of the constraint violations, has been reduced to 1 in the tableau above. Phase 1 is not yet complete though: there is still a negative coefficient in the phase 1 objective function row. Note that the phase 2 objective has also been updated to reflect its value at this new (infeasible) cornerpoint.

basic variable	eqn. no.	W	Z	x_1	x_2	s_1	s_2	a_1	RHS	MRT
W	ph1	-1	0	0	0	0	0	1	0	*never*
Z	ph2	0	1	0	0	0	-5	15	45	*never*
s_1	1	0	0	0	0	1	1	-1	1	
x_2	2	0	0	0	1	0	1	0	3	
x_1	3	0	0	1	0	0	-1	1	1	

Phase 1 is now complete: no constraints are violated, and we are at a feasible cornerpoint for the original problem. Now it is time to discard the phase 1 objective function, the W column, and all of the artificial variables (just a_1 in this case), which results in the tableau below. Note that the phase 2 objective function, which has been carried through all of the calculations so far, is updated, in proper form, and ready to go. And because there is a negative coefficient in the phase 2 objective function row, the iterations must continue.

basic variable	eqn. no.	Z	x_1	x_2	s_1	s_2	RHS	MRT
Z	ph2	1	0	0	0	-5	45	*never*
s_1	1	0	0	0	1	1	1	1/1=1
x_2	2	0	0	1	0	1	3	3/1=3
x_1	3	0	1	0	0	-1	1	no limit

basic variable	eqn. no.	Z	x_1	x_2	s_1	s_2	RHS	MRT
Z	ph2	1	0	0	5	0	50	*never*
s_2	1	0	0	0	1	1	1	
x_2	2	0	0	1	-1	0	2	
x_1	3	0	1	0	1	0	2	

After this final phase 2 iteration, the solution is complete: there are no negative coefficients remaining in the phase 2 objective function. The final solution is (x_1,x_2,s_1,s_2)

Recognizing Infeasible LP

If there is no feasible solution, it is difficult to construct linear programs

$$x \leq 8 \quad \& \quad x \geq 10$$

If the phase 1 LP terminates, W is still positive, then not all of the constraint violations have been eliminated. This means the LP is infeasible.

One or more of the artificial variables could not be forced to zero and for feasibility, all of the artificial variables must be forced to zero.

Tools (*IIS*)

The Big-M Method

It is an alternative to the two-phase method that we described above

Phase 1 Objective Function   Phase 2 Objective Function

↓

Single Objective Function

$$\text{maximize } Z = 15x_1 + 10x_2 - Ma_1$$

If M is a large positive number then a straightforward solution of the problem will drive a1 to zero, because it has such a reducing effect on Z.

Greater-than-or-Equal Constraints

The solution to this problem is to convert ≥

To an equality constraint by including a surplus variable

$$3x_1 + 5x_2 \geq 20 \;\Rightarrow\; 3x_1 + 5x_2 - s_1 = 20.$$

A surplus variable cannot be used as the basic variable for the constraint it appears in because its coefficient is -1, and we need +1 for the coefficients of basic variables

$$3x_1 + 5x_2 - s_1 + a_1 = 20$$

So there will be *two* variables added to every ≥ constraint: one surplus variable (with a coefficient of –1) and one artificial variable (with a coefficient of +1). Any artificial variables added to ≥ constraints will be driven to zero during phase 1.

Thank You!