1. Define a map $F : C_{[0,1]} \to \mathbb{R}$ by letting

$$F(f) = \int_0^1 |f(x)|^{3/2} \, dx.$$ 

For the following choices of metrics, determine whether $F$ is continuous:

(a) $\rho(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$

(b) $\rho_1(f,g) = \int_0^1 |f(x) - g(x)| \, dx$

(c) $\rho_2(f,g) = \left( \int_0^1 |f(x) - g(x)|^2 \, dx \right)^{1/2}$

2. Prove Hölder’s inequality for integrals. Namely if $f, g \in C_{[0,1]}$, and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^1 f(x)g(x) \, dx \leq \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \left( \int_0^1 |g(x)|^q \, dx \right)^{1/q}.$$ 

3. Let $(X, \rho)$ be a metric space.

(a) Fix a point $x_0$ in $X$ and define the function $f : X \to \mathbb{R}$ by letting $f(x) = \rho(x, x_0)$ for all $x$. Prove that $f$ is continuous.

(b) Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in $X$ converging to $x$ and $y$ respectively. Prove that

$$\lim_{n \to \infty} \rho(x_n, y_n) = \rho(x, y).$$

4. Let $f : X \to Y$ be a map between metric spaces, and let $x \in X$ be a point. Prove that $f$ is continuous at $x$, if and only if whenever $\{x_n\}$ is a sequence converging to $x$, the sequence $\{f(x_n)\}$ converges to $f(x)$.

5. Recall the metrics $\rho_2$ and $\rho_\infty$ on $\mathbb{R}^n$ that we defined by

$$\rho_2(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2}$$

$$\rho_\infty(x, y) = \max_{i=1,\ldots,n} |x_i - y_i|.$$ 

Show that $(\mathbb{R}^2, \rho_2)$ and $(\mathbb{R}^2, \rho_\infty)$ are not isometric. (Note that it is not enough to consider the identity map.)
6. Suppose that \( f_n \in C_{[0,1]} \) is a sequence converging to a function \( f : [0, 1] \to \mathbb{R} \) uniformly. I.e. we have
\[
\lim_{n \to \infty} \sup_{x \in [0, 1]} |f_n(x) - f(x)| = 0.
\]
Prove that \( f \) is continuous.

7. Recall the metric space \( m \), whose elements are the bounded infinite sequences of real numbers, with the metric
\[
\rho(x, y) = \sup_{i=1,2,...} |x_i - y_i|.
\]
Prove that \( m \) is complete.

8. Let \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \) be a nested sequence of non-empty closed sets in a complete metric space \((\mathbb{R}, \rho)\).

(a) Give an example to show that the intersection \( \bigcap_{k \geq 1} A_k \) may be empty.

(b) Suppose that the sets \( A_k \) are bounded, i.e. we can define the diameters
\[
d(A_k) = \sup_{x, y \in A_k} \rho(x, y).
\]
Prove that if \( \lim_{k \to \infty} d(A_k) = 0 \), then
\[
\bigcap_{k \geq 1} A_k \neq \emptyset.
\]

(c) What can we say when the sets \( A_k \) are bounded, but the diameters do not converge to zero? Is the intersection \( \bigcap_{k \geq 1} A_k \) non-empty?