Honors Analysis - Homework 5

1. Let $V$ be a Banach space, and $W \subset V$ a closed subspace. Show that the quotient space $V/W$ is also complete, i.e. a Banach space.

2. Suppose that $V$ is a normed linear space, and $f : V \to \mathbb{R}$ is a linear functional (which may not be continuous). Show that $f$ is continuous if and only if the kernel $\text{Ker} f$ is closed.

3. Suppose that $F : C_{[0,1]} \to \mathbb{R}$ is a linear functional, which satisfies the property that $F(g) \geq 0$ whenever $g \in C_{[0,1]}$ is a non-negative function (i.e. if $g(x) \geq 0$ for all $x \in [0,1]$). Prove that $F$ is a continuous linear functional, with respect to the sup norm on $C_{[0,1]}$.

4. Let $V$ be a normed linear space.
   (a) Prove that every finite dimensional subspace of $V$ is closed.
   (b) For two subspaces $A, B \subset V$ we define the sum
       $$A + B = \{ x + y | x \in A, y \in B \}.$$  
       Prove that if $A$ is a closed subspace, and $B$ is finite dimensional, then $A + B$ is a closed subspace of $V$.

5. Let $V$ be a normed linear space.
   (a) Suppose that $W \subset V$ is a closed subspace. Show that there exists an element $v \in V \setminus W$ such that $\|v\| = 1$ and 
       $$\|v - w\| > \frac{1}{2}$$  
       for every $w \in W$.
   (b) Prove that if the closed unit ball $\{ x \in V | \|x\| \leq 1 \}$ is compact, then $V$ is finite dimensional.

6. Let $(X, \rho)$ be a complete metric space, and let $S$ be the set of all non-empty compact subsets of $X$. For $A, B \in S$, define the distance
   $$d(A, B) = \max \{ \sup_{x \in A} \inf_{y \in B} \rho(x, y), \sup_{y \in B} \inf_{x \in A} \rho(x, y) \}.$$  
In other words $d(A, B) \leq k$ means that for every point $x \in A$ there is a point $y \in B$ with $d(x, y) \leq k$ and vice versa, i.e. for every point $y \in B$ there is an $x \in A$ with $d(x, y) \leq k$.
   (a) Show that $(S, d)$ is complete.
   (b) Assuming that $(X, \rho)$ is compact, prove that $(S, d)$ is totally bounded (so, combined with (a), this means $(S, d)$ is compact).