**A list of statements/theorems that you should be able to prove, together with the main idea of the proof for some of them.**

1. Let $A, A_k$ be elementary subsets of $[0,1]\times[0,1]$, such that
$$A \subset \bigcup_{k=1}^{\infty} A_k.$$
Then
$$\tilde{m}(A) \leqslant \sum_{k=1}^{\infty} \tilde{m}(A_k).$$
*[Replace $A$ by a slightly smaller closed set, and enlarge each $A_k$ a bit to get open sets. Then use compactness of the new $A$ to reduce to a finite union.]*

2. For every $A \subset [0,1]\times[0,1]$ we have $\mu_*(A) \leqslant \mu^*(A)$.
*[Otherwise we would have $\mu^*(A) + \mu^*(E \setminus A) < 1$, where $E = [0,1]\times[0,1]$, and we could get too small a cover of $E$, contradicting the previous theorem.]*

3. Suppose that $A, A_k \subset [0,1]\times[0,1]$, and $A \subset \bigcup_{k=1}^{\infty} A_k$. Then
$$\mu^*(A) \leqslant \sum_{k=1}^{\infty} \mu^*(A_k).$$
*[By definition of $\mu^*$ each $A_k$ can be covered by rectangles whose areas sum to slightly more than $\mu^*(A_k)$. The collection of all these rectangles cover $A$, giving an upper bound on $\mu^*(A)$.]*

4. If $A \subset [0,1]\times[0,1]$ is an elementary set, then $A$ is measurable and $\mu(A) = \tilde{m}(A)$.
*[$\tilde{m}(A) \leqslant \mu^*(A)$ follows from first theorem, and $\mu^*(A) \leqslant \tilde{m}(A)$ follows from writing $A$ as a disjoint union of rectangles. Apply the same also to the complement of $A$.]*

5. The union, intersection, difference and symmetric difference of two measurable subsets of $[0,1]\times[0,1]$ is measurable.
*[Use the fact that $A$ is measurable if and only if for every $\epsilon > 0$ there is an elementary set $B$ such that $\mu^*(A \triangle B) < \epsilon$.]*

6. Suppose that $A_1, A_2$ are disjoint measurable subsets of $[0,1]\times[0,1]$. Then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.
*[Approximate $A_1$ and $A_2$ with elementary sets, and use the additivity of $\tilde{m}$. Note that the approximating sets may not be disjoint. ]*

7. The union of a countable collection of measurable sets is measurable.
*[It's enough to do this for disjoint unions. Approximate the countable union with a finite union, and approximate the finite union with an elementary set.]*

8. If $A = \bigcup_k A_k$ is a disjoint union of a countable collection of measurable sets, then
$$\mu(A) = \sum_{k=1}^{\infty} \mu(A_k).$$
*[We know that $\mu(A) \leqslant \sum \mu(A_k)$. For the converse inequality use finite additivity.]*

9. Let $f_n : X \to \mathbf{R}$ be measurable, such that the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x$. Then $f$ is measurable.
*[Express $f^{-1}(-\infty, c)$ in terms of sets of the form $f_n^{-1}(-\infty, d)$ using countably many unions / intersections]*

10. If $f : X \to \mathbf{R}$ is measurable, and $g : \mathbf{R} \to \mathbf{R}$ is continuous, then $g \circ f$ is measurable.

11. A function $f : X \to \mathbf{R}$ is measurable if and only if $f$ is a uniform limit of simple functions.
*[For any $k > 1$ define a simple function $g : X \to \mathbf{R}$ by letting $g(x) = \frac{m}{k}$ if $\frac{m}{k} \leqslant f(x) < \frac{m+1}{k}$ for an integer $m$. Then $g$ is simple, and $|f - g| \leqslant \frac{1}{k}$]*

12. Suppose $f, g : X \to \mathbf{R}$ are measurable and $c \in \mathbf{R}$. Then $f + g$, $cf$, $fg$, $f/g$ are measurable if $g$ is nowhere vanishing in the case of $f/g$.

13. Let $f, g : [0, 1] \to \mathbf{R}$ be continuous such that $f(x) = g(x)$ for almost every $x$ (with respect to Lebesgue measure). Then $f(x) = g(x)$ for all $x$.

14. (Egorov's theorem) Let $f_n : X \to \mathbf{R}$ be a sequence of measurable functions, converging almost everywhere to $f : X \to \mathbf{R}$. For any $\delta > 0$ there exists a set $Y \subset X$ such that $\mu(X \setminus Y) < \delta$ and $f_n \to f$ uniformly on $Y$.
*[For $m, n > 0$ let $E_n^m$ be the set of $x$ such that $|f_i(x) - f(x)| < 1/m$ for all $i > n$. For almost every $x$ we have $x \in \bigcup_n E_n^m$. Use this to show that there is an $N_m$ such that $\mu(E_{N_m}^m) > \mu(X) - 2^{-m}\delta$. Finally define $Y = \bigcap_m E_{N_m}^m$.]*

15. Suppose that $\phi : A \to \mathbf{R}$ is integrable and $f : A \to \mathbf{R}$ satisfies $|f(x)| \leqslant \phi(x)$ for all $x \in A$. Then $f$ is integrable and
$$\left| \int_A f(x)\, \mu \right| \leqslant \int_A \phi(x)\, \mu.$$
*[First assume that $f, \phi$ are simple functions. Then use approximation to extend to the general case.]*

16. (Chebyshev's inequality) If $f : A \to \mathbf{R}$ is integrable and $f(x) \geqslant 0$ for all $x \in A$, then
$$\mu\{x\,;\, x \in A,\ f(x) \geqslant c\} \leqslant \frac{1}{c} \int_A f(x)\, d\mu,$$
for all $c > 0$.
*[Split the integral over $A$ into two parts, over the sets where $f \geqslant c$ and where $f < c$.]*

17. Let $f : A \to \mathbf{R}$ be integrable. For any $\epsilon > 0$ there is a $\delta > 0$ such that
$$\left| \int_E f(x)\, d\mu \right| \leqslant \epsilon,$$
whenever $E \subset A$ satisfies $\mu(E) < \delta$.
*[First find an $N$ such that the integral of $|f|$ on the set where $|f| > N$ is at most $\epsilon/2$. Then the integral of $|f|$ on a set $E$ will be at most $\epsilon/2 + N\mu(E)$.]*

18. (Bounded convergence theorem) Let $f_n \to f$ almost everywhere on $A$, and let $\phi : A \to \mathbf{R}$ be an integrable function such that $|f_n(x)| \leqslant \phi(x)$ for almost every $x \in A$. Then $f$ is integrable, and

$$\int_A f(x)\, d\mu = \lim_{n\to\infty} \int_A f_n(x)\, d\mu.$$

*[By Egorov's theorem $f_n \to f$ uniformly outside of a small set $C \subset A$. If $C$ has sufficiently small measure, then we can make the integral of $\phi$ on $C$ as small as needed.]*

19. (Monotone convergence theorem) Suppose that $f_1(x) \leqslant f_2(x) \leqslant \dots$ for all $x \in A$, each $f_n$ is integrable, and

$$\int_A f_n\, d\mu \leqslant M,$$

for some constant $M$. Then $f(x) = \lim_{n\to\infty} f_n(x)$ is defined almost everywhere on $A$, $f$ is integrable, and

$$\int_A f\, d\mu = \lim_{n\to\infty} \int_A f_n\, d\mu.$$

*[First show that for almost every $x$ the sequence $f_n(x)$ is bounded, by looking at the measure of the set of $x$ for which $f_n(x) > C$ for large $C$. Now you can define $f(x)$ almost everywhere. To control its integral, find a simple function bigger than $f$, and show that it's integrable, then apply the bounded convergence theorem.]*

20. (Fatou's theorem) Let $f_n \geqslant 0$ be integrableon $A$, such that for some $M > 0$ we have

$$\int_A f_n\, d\mu \leqslant M,$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in A$. Then $f$ is integrable, and

$$\int_A f\, d\mu \leqslant M.$$

*[ Let $\phi_n = \inf_{k\geqslant n} f_k$, and apply the monotone convergence theorem to $\phi_n$.]*

21. The space $L^1(X, \mu)$ is complete.
*[If $f_n$ is a Cauchy sequence in $L^1$, then a subsequence $f_{n_k}$ satisfies $\|f_{n_{k+1}} - f_{n_k}\|_1 < 2^{-k}$. Construct the limit of this subsequence as a telescoping sum, and use the monotone convergence theorem to give the existence of the limit.]*

22. The space $L^2(X, \mu)$ is complete.
*[Show that a Cauchy sequence in $L^2$ is also a Cauchy sequence in $L^1$, and then use the completeness of $L^1$.]*

23. (Lusin's Theorem) Let $f : [a, b] \to \mathbf{R}$ be measurable, with respect to the Lebesgue measure. For every $\epsilon > 0$ there is a set $E \subset [a, b]$ with $\mu([a, b] \setminus E) < \epsilon$ such that the restriction of $f$ to $E$ is continuous.
*[Use the density of continuous functions in $L^1$ together with Egorov's theorem]*

24. (Riesz representation theorem) For every bounded linear functional $f : H \to \mathbf{C}$ on a Hilbert space $H$, there is an element $y \in H$ such that

$$f(x) = \langle x, y \rangle, \text{ for all } x \in H.$$

*[Let $y = \|\tilde{y}\|^{-2}\tilde{y}$, where $\tilde{y}$ is the closest point to the origin in $f^{-1}(0)$.]*

25. If $A : E \to F$ is a bounded linear operator between Banach spaces, then the adjoint $A^*$ is bounded, and $\|A^*\| = \|A\|$.
*[To get a lower bound on $\|A^*\|$, you need to use the Hahn-Banach theorem to write $\|Ax\| = |g(Ax)|$ for some $g \in F^*$.]*

26. The set of invertible elements in a Banach algebra with unit is open.
*[Use that if $\|a\| < 1$, then $e - a$ is invertible.]*

27. Any maximal ideal in a Banach algebra with unit is closed.
*[Use the fact that the invertible elements form an open set to show that the closure of a proper ideal is closed.]*

28. If $a \in A$ is an element in a Banach algebra with unit, then the spectral radius $\nu(a)$ satisfies $\nu(a) \leqslant \|a\|$.
*[Use that $e - x$ is invertible if $\|x\| < 1$.]*

29. If $A$ is a Banach algebra where every non-zero element is invertible, then $A \cong \mathbf{C}$.
*[The spectrum of every element is non-empty.]*

30. If $a \in A$ is normal in a $C^*$-algebra $A$, then $\nu(a) = \|a\|$.
*[Use the spectral radius formula.]*

31. If $A$ is a commutative Banach algebra with unit, then there is a bijection between maximal ideals of $A$ and non-zero homomorphisms $A \to \mathbf{C}$.
*[The bijection is given by identifying a homomorphism with its kernel.]*

32. For a commutative Banach algebra $A$ with unit, the spectrum $\sigma(a)$ of any element is the range of its Gelfand transform $\hat{a}$.
*[Use that an element $x \in A$ is invertible if and only if $\phi(x) \neq 0$ for all non-zero homomorphisms $\phi$.]*