Homework 10, due 12/5

1. Let $X$ be a compact Riemann surface, and for any $(1,0)$-form $\theta \in \Omega^{1,0}_X$, define the norm $\|\theta\|$ by

$$\|\theta\|^2 = i \int_X \theta \wedge \overline{\theta}.$$ 

From the previous homework we know that this is a non-negative real number, which vanishes only if $\theta = 0$. Denote by $[\theta]$ the equivalence class of $\theta$ in $\Omega^{1,0}_X/(\text{im } \partial)$.

Show that if $\alpha \in [\theta]$ has minimal norm among the elements in the class $[\theta]$, then $\overline{\partial} \alpha = 0$, i.e. $\alpha$ is a holomorphic one-form. (Note that this gives another approach to proving the isomorphism $H^0,1 = H^{1,0}$ from class.)

2. Show that there is no non-constant holomorphic map $f : \mathbb{P}^1 \to X$, where $X = \mathbb{C}/\Lambda$ is a complex torus. \{Hint: use $f$ to define a holomorphic one-form on $\mathbb{P}^1$\}

3. Let $\Lambda_1 = \{m_1w_1 + m_2w_2 : m_1, m_2 \in \mathbb{Z}\}$ for some $w_1, w_2 \in \mathbb{C}$ satisfying $\text{Im}(w_1/w_2) > 0$. Similarly define $\Lambda_2$ using $u_1, u_2 \in \mathbb{C}$ satisfying $\text{Im}(u_1/u_2) > 0$.

Show that the torus $X/\Lambda_1$ is biholomorphic to $X/\Lambda_2$ if and only if $\Lambda_1 = c\Lambda_2$ for some $c \in \mathbb{C}$. \{Hint: think about how a biholomorphism relates the holomorphic one-forms.\}