Homework 4, due 9/23

1. Let \( f : \mathbb{C} \to \mathbb{C} \) be holomorphic, and not constant. Show that \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \).

2. Let \( f \) be a meromorphic function on \( \mathbb{C} \).
   
   (i) Suppose that there exist \( k, C > 0 \) such that \( |f(z)| \leq C|z|^k \) for all \( |z| > C \). Prove that \( f \) is a rational function, i.e. there are polynomials \( p, q \) such that \( f = p/q \).
   
   (ii) Suppose that the function \( g(w) = f(1/w) \) is also meromorphic on \( \mathbb{C} \). Prove that \( f \) is a rational function.

3. Find the Laurent series of the function

\[
f(z) = \frac{1}{1-z^2},
\]

around the point \( z = -1 \). Where does the series converge?

4. Let \( f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) be the meromorphic function defined by

\[
f(z) = \frac{1 - \cos z}{z^5}.
\]

Find \( \text{ord}_0(f) \).

5. Prove that the function \( f(z) = \sin(1/z) \) has an essential singularity at \( z = 0 \).

6. Let \( f : D(0, 1) \to \mathbb{C} \) be holomorphic such that \( f(0) = 0 \). Show that there is an integer \( m \), an \( r > 0 \), and a holomorphic \( g : D(0, r) \to \mathbb{C} \) with \( g(0) \neq 0 \) such that for \( z \in D(0, r) \) we have

\[
f(z) = \left[zg(z)\right]^m.
\]

7. Consider the improper integral

\[
I = \lim_{R \to \infty} \int_0^R e^{ix^2} \, dx
\]

on the positive real axis. Prove that

\[
I = \lim_{R \to \infty} \int_{\gamma_R} e^{iz^2} \, dz,
\]

where \( \gamma_R \) is the line segment \( \gamma_R(t) = te^{i\theta} \) for any \( \theta \in (0, \pi/2) \), with \( t \in [0, R] \). Deduce that

\[
I = e^{\pi i/4} \int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2} e^{\pi i/4}.
\]