Homework 8, due 11/11

1. Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces, such that $X$ is compact. Show that then $Y$ is compact and $f$ is surjective.

2. Let $f : X \to Y$ be a holomorphic map of Riemann surfaces, and let $x \in X$, $y = f(x) \in Y$. Show that we can find charts $(U, \phi)$ and $(V, \psi)$ on $X, Y$ containing $x, y$, such that

$$
\psi \circ f \circ \phi^{-1}(z) = z^k,
$$

for an integer $k$ wherever the composition is defined.

3. Let $w_1, w_2 \in \mathbb{C}$ be given, such that $\text{Im}(w_1/w_2) > 0$. Define the lattice $\Lambda = \{m_1w_1 + m_2w_2 : m_1, m_2 \in \mathbb{Z}\}$, and consider the complex torus $X = \mathbb{C}/\Lambda$.

   (a) Show that the 1-form $dz$ on $\mathbb{C}$ defines a holomorphic 1-form $\alpha$ on $X$.

   (b) Consider the closed curve $\gamma(t) = tw_1$ for $t \in [0, 1]$ on $X$, and compute

$$
\int_{\gamma} \alpha.
$$

   (c) Show that all holomorphic 1-forms on $X$ are of the form $c\alpha$ for some $c \in \mathbb{C}$.

4. Consider the complex torus $X$ from the previous problem. Define the function

$$
g(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right).
$$

Show that $g$ defines a meromorphic function on $\mathbb{C}$, which is periodic with respect to the lattice $\Lambda$, and so $g$ defines a meromorphic function on $X$.

5. Let $f$ be a non-constant meromorphic function on a compact Riemann surface $X$. Show that

$$
\sum_p \text{ord}_p(f) = 0,
$$

where the sum is finite since $\text{ord}_p(f)$ is only non-zero at poles or zeros of $f$. 