Homework 1, due 9/3

1. For each \( c \in D(0,1) \) define the transformation \( L_c \) by

\[
L_c(z) = \frac{z-c}{1-ar{c}z}.
\]

Prove that \( L_c \) maps the unit disk \( D(0,1) \) onto the unit disk, and the unit circle \( S^1 = \{ z : |z| = 1 \} \) onto the unit circle.

2. If \( f : \Omega \to \mathbb{C} \) is holomorphic on a connected open set \( \Omega \subset \mathbb{C} \), prove the following:

(i) If \( f'(z) = 0 \) for all \( z \in \Omega \), then \( f \) is constant.

(ii) If there exists \( c \in \mathbb{C} \) such that \( f(z) = c \cdot \overline{f(z)} \) for every \( z \in \Omega \), then \( f \) is constant.

(iii) If \( f(\Omega) \subset \mathbb{R} \), then \( f \) is constant.

3. Suppose that \( f : \Omega \to \mathbb{C} \) has continuous first partial derivatives, and the real and imaginary parts of \( f \) satisfy the Cauchy-Riemann equations (so in particular \( f \) is holomorphic). Let \( \gamma \) be the boundary of a smooth domain in \( \Omega \) oriented positively. Show, using Green’s Theorem in the plane, that

\[
\int_{\gamma} f(z) \, dz = 0.
\]

4. Let \(|a| < r < |b|\). Show that

\[
\int_{\gamma} \frac{1}{(z-a)(z-b)} \, dz = \frac{2\pi i}{a-b},
\]

where \( \gamma \) is the circle of radius \( r \) centered at the origin (oriented counterclockwise).