Homework 2, due 9/17
Only your five best solutions will count towards your grade.

1. Let $C = \partial D(0, 2)$ denote the circle of radius 2 around the origin, oriented positively. Compute the following integrals:
   (a) $\int_C \frac{1}{z^2 - 1} \, dz$
   (b) $\int_C \frac{e^z}{(z - 1)^n} \, dz$
   for all integers $n \geq 0$.

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic, and for some $d, C > 0$ we have
   $$|f(z)| < C(1 + |z|^d) \quad \text{for all } z \in \mathbb{C}.$$  
   (a) Show that if $d < 1$, then $f$ is constant.
   (b) Show that if $d \leq k$, then $f$ is a polynomial of degree at most $k$.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that the image $f(\mathbb{C})$ is dense in $\mathbb{C}$.

4. Let $D = D(0, 1) \subset \mathbb{C}$ be the open unit disk, and $u : D \to \mathbb{R}$ be harmonic, i.e.
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$  
   Here $x, y$ are the real and imaginary parts of $z$, and you may assume that $u$ has continuous second partial derivatives. Show that there is a holomorphic function $f : D \to \mathbb{C}$ such that $u = \text{Re}(f)$. (Hint: consider what $f'$ would have to be.)

5. Suppose that $\gamma : [0, 1] \to \mathbb{C}$ is a smooth curve parametrizing the boundary $\partial \Omega$ of an open set $\Omega \subset \mathbb{C}$ oriented positively. Show that the area $A(\Omega)$ is given by
   $$A(\Omega) = \frac{1}{2i} \int_\gamma \bar{z} \, dz.$$  

6. Prove that if $N > 0$ is an integer and $f$ is holomorphic on $D(0, 2)$ with
   $$|f^{(N)}(0)| = N! \sup\{|f(z)| : |z| = 1\},$$
   then $f(z) = cz^N$ for some $c \in \mathbb{C}$.

7. Suppose that $f(z) = \sum_{n \geq 0} c_n z^n$ defines a holomorphic function on $D(0, 1)$, such that $f(z) \in \mathbb{R}$ for all $z \in D(0, 1) \cap \mathbb{R}$. Show that $c_n \in \mathbb{R}$ for all $n$. 

8. Consider the improper integral

\[ I = \lim_{R \to \infty} \int_0^R e^{ix^2} \, dx \]

on the positive real axis. Prove that

\[ I = \lim_{R \to \infty} \int_{\gamma_R} e^{iz^2} \, dz, \]

where \( \gamma_R \) is the line segment \( \gamma_R(t) = te^{i\theta} \) for any \( \theta \in (0, \pi/2) \), with \( t \in [0, R] \). Using that \( \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi}/2 \), deduce that

\[ I = \frac{\sqrt{\pi}}{2} e^{\pi i/4}. \]