Homework 3, due 9/24
Only your four best solutions will count towards your grade.

1. Suppose that $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic, and $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that $f = cg$ for some $c \in \mathbb{C}$.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and injective. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, with $a \neq 0$.

3. Compute the integral
   $$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} \, dx.$$  

4. Compute the integral
   $$\int_{0}^{\infty} \frac{1}{x^3 + 1} \, dx. 
   \text{Hint: consider the integral of } \log z/(z^3 + 1) \text{ along a suitable contour.}$$

5. Use the residue formula to prove that
   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$
   by considering the integral of $f(z) = \cot(\pi z)/z^2$ around large circles centered at the origin.

6. Suppose that $f$ is holomorphic on the punctured disk $D(0, 1) \setminus \{0\}$, and that for some $A, \epsilon > 0$ we have
   $$|f(z)| < A|z|^{-1+\epsilon},$$
   for all $z \in D(0, 1) \setminus \{0\}$. Show that 0 is a removable singularity of $f$.

7. Prove that the function $f(z) = \sin(1/z)$ has an essential singularity at $z = 0$. 

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