Homework 5, due 10/29

1. Let $f : D \to D$ be holomorphic on the unit disk $D$, such that $f(0) = 0$.
   
   (a) Prove that $|f(z) + f(-z)| \leq 2|z|^2$ for all $z \in D$.

   (b) Suppose that $|f(z_0) + f(-z_0)| = 2|z_0|^2$ for some $z_0 \neq 0$. Show that then $f(z) = e^{i\theta}z^2$ for some constant $\theta \in \mathbb{R}$, for all $z \in D$.

2. Let $A \subset \mathbb{C}$ denote the half-disk $A = \{z : |z| < 1, \text{Re } z > 0\}$, and $B$ denote the quarter plane $B = \{z : \text{Re } z, \text{Im } z > 0\}$.
   
   (a) Find a biholomorphism $f : A \to B$.

   (b) Find a biholomorphism $g : B \to D(0,1)$ to the unit disk.

3. Does there exist a surjective holomorphic map from the unit disk to $\mathbb{C}$?

4. Show that the map $z \mapsto \int_1^z \frac{dw}{(1 - w^n)^{2/n}}$ is a biholomorphism from the unit disk to the interior of a regular $n$-gon.