Homework 9, due 12/5
Only your four best solutions will count towards your grade.

1. Suppose that \( \alpha \) is a \((1,0)\)-form on a compact Riemann surface \( X \).
   (a) If in a local holomorphic chart \( \alpha = \alpha_z dz \), define \( \bar{\alpha} = \alpha_{\bar{z}} d\bar{z} \). Show that \( \bar{\alpha} \) defines a \((0,1)\)-form on \( X \), i.e. check that the coordinate representations of \( \bar{\alpha} \) satisfy the right compatibility condition.
   (b) Show that
   \[
   \int_X i^2 \alpha \wedge \bar{\alpha} \geq 0,
   \]
   with equality only if \( \alpha = 0 \).
   (c) Suppose that \( f : X \to \mathbb{C} \) satisfies \( \partial \bar{\partial} f = 0 \) (and \( X \) is compact). Show that \( f \) is constant, by considering the integral of \( \partial f \wedge \bar{\partial} f \) and using Stokes' Theorem.

2. Let \( X \) be a compact Riemann surface, and for any \((1,0)\)-form \( \theta \in \Omega^{1,0}_X \), define the norm \( \| \theta \| \) by
   \[
   \| \theta \|^2 = i \int_X \theta \wedge \bar{\theta}.
   \]
   From the previous question we know that this is a non-negative real number, which vanishes only if \( \theta = 0 \). Denote by \([\theta]\) the equivalence class of \( \theta \) in \( \Omega^{1,0}_X/(\text{im } \partial) \).
   Show that if \( \alpha \in [\theta] \) has minimal norm among the elements in the class \([\theta]\), then \( \bar{\partial} \alpha = 0 \), i.e. \( \alpha \) is a holomorphic one-form. (Note that this gives another approach to proving the isomorphism \( H^{0,1} = \overline{H^{1,0}} \) from class.)

3. Let \( \alpha \) be a \(2\)-form supported in a chart \( U \) on a Riemann surface. Suppose that \( z, w \) are two local coordinates on \( U \), and \( \alpha = f(z)dz \wedge d\bar{z} \) and \( \alpha = g(w)dw \wedge d\bar{w} \) are the expressions of \( \alpha \) in these coordinates. Show that the integral \( \int_U \alpha \) defined in class is independent of the coordinate representation chosen for \( \alpha \).

4. (a) Let \( \alpha \) be any meromorphic one-form on \( \mathbb{P}^1 \). Show that
   \[
   \sum_{p \in \mathbb{P}^1} \text{ord}_p \alpha = -2.
   \]
   Hint: show that \( \alpha = f dz \) for a meromorphic function \( f \).
   (b) Let \( p_1, \ldots, p_k \in \mathbb{P}^1 \), and \( a_1, \ldots, a_k \in \mathbb{Z} \) satisfy \( \sum_i a_i = -2 \). Can you find a meromorphic one-form \( \alpha \) on \( \mathbb{P}^1 \) such that \( \text{ord}_{p_i} \alpha = a_i \) for each \( i \), and \( \text{ord}_p \alpha = 0 \) for all other \( p \)?

5. Consider the one-form \( \alpha = \bar{z} dz \) on \( \mathbb{C} \).
   (a) Does there exist a function \( f : \mathbb{C} \to \mathbb{C} \) such that \( \alpha = df \)?
(b) Does there exist $f : \mathbb{C} \to \mathbb{C}$ such that $\alpha = \partial f$?

6. In class we showed that $\dim H^{1,0}_X \leq g$, where $g$ is the genus of the compact Riemann surface $X$. Let

$$H^1(X, \mathbb{R}) = \frac{\ker(d : \Omega^1(X) \to \Omega^2(X))}{d\Omega^0(X)}$$

denote the De Rham cohomology of $X$. Show that $\dim_{\mathbb{R}} H^{1,0}_X = \dim_{\mathbb{R}} H^1(X, \mathbb{R})$ by showing that the map $H^{1,0}_X \to H^1(X, \mathbb{R})$ given by $\alpha \mapsto \alpha + \bar{\alpha}$ is a (real linear) isomorphism. This can be used to show that $\dim H^{1,0}_X = g$. 

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