Homework 8, due 4/17
Only your four best solutions will count towards your grade.

1. Let $E \to X$ be a complex vector bundle, with a Hermitian metric $h$, and let $\nabla$ be a connection on $E$. We can view $h$ as a section of the bundle $E^* \otimes E^*$ (or equivalently as a bilinear map $E \times E \to \mathbb{C}$), and $\nabla$ induces a natural connection on this bundle. Show that $\nabla$ is compatible with $h$ if and only if $\nabla h = 0$.

2. Let $L \to X$ be a complex line bundle over $X$, with a connection $\nabla$. Then the curvature $F_{\nabla}$ is a (complex valued) closed 2-form on $X$. Show that every closed 2-form cohomologous to $F_{\nabla}$ is the curvature of a connection on $L$.

3. Let $(L,h) \to X$ be a Hermitian line bundle, and $\nabla$ a Hermitian connection on it. Show that the curvature $F_{\nabla}$ is a purely imaginary 2-form.

4. Show that the map
$$\nabla^2 : \mathcal{A}^k(E) \to \mathcal{A}^{k+2}(E)$$
is given by $F_{\nabla}$, acting on the form part by the exterior product, and on the $E$ part through its endomorphism component. I.e. if locally $F_{\nabla} = \sum_{i,j} F_{ij} dx^i \wedge dx^j$ for endomorphisms $F_{ij}$, and $\alpha \otimes s$ is a section of $\mathcal{A}^k(E)$, where $\alpha$ is a $k$-form and $s$ a section of $E$, then
$$\nabla^2 s = \sum_{i,j} dx^i \wedge dx^j \wedge \alpha \otimes F_{ij}(s).$$

5. Let $X$ be a compact complex manifold and $D \subset X$ a codimension-one complex submanifold. Define the line bundle $L$ as in Problem 2 from homework set 5 (usually denoted by $\mathcal{O}(D)$), and let $s$ be a global holomorphic section of $L$ whose zero set is $D$. Choose a Hermitian metric $h$ on $L$, let $\nabla$ be the Chern connection, and $F_{\nabla}$ its curvature. Show that for any closed form $\alpha$ we have
$$\int_X F_{\nabla} \wedge \alpha = \lim_{\epsilon \to 0} \int_{\partial \mathcal{D}_\epsilon} \partial \log |s|^2_h \wedge \alpha,$$
where $\mathcal{D}_\epsilon = \{x : |s(x)|_h < \epsilon\}$ is a small neighborhood of $D$, and $|s|^2_h = h(s,s)$.

6. With the same notation as in the previous question, show that
$$\frac{\sqrt{-1}}{2\pi} \int_X F_{\nabla} \wedge \alpha = \int_D \alpha,$$
i.e. the first Chern class $c_1(L)$ represents the Poincaré dual of $D$. 