Homework 9, due 5/1
Only your four best solutions will count towards your grade.

1. Let $E$ be a complex vector bundle over a complex manifold $X$, and $\nabla$ a connection on $E$. Show that the trace $\text{tr} F_\nabla$ of the curvature defines a closed two-form on $X$.

2. In the setting of the previous question, show that if $\nabla'$ is another connection on $E$, then $[\text{tr} F_{\nabla'}] = [\text{tr} F_\nabla]$ in $H^2(X, \mathbb{C})$.

3. Let $L$ be a holomorphic line bundle over a complex manifold $X$. Suppose that we have a sheaf homomorphism

$$D : L \to \Omega_X \otimes L,$$

satisfying the Leibniz rule $D(f \cdot s) = \partial f \otimes s + f \cdot D(s)$ for local holomorphic functions $f$ and holomorphic sections $s$ of $L$. Here $\Omega_X$ denotes the sheaf of holomorphic $(1,0)$-forms on $X$, and we are using $L$ to denote the sheaf of holomorphic sections of $L$.

Show that $D$ can be extended to a connection

$$\nabla : \mathcal{A}^0(L) \to \mathcal{A}^1(L)$$

on $L$ such that $\nabla s = Ds$ for holomorphic sections $s$, and the curvature of $\nabla$ is a holomorphic $(2,0)$-form on $X$.

4. In the setting of the previous question, if in addition $X$ is a compact Kähler manifold, show that the curvature of $\nabla$ vanishes.

5. Let $(E, h)$ be a Hermitian vector bundle over a complex manifold $X$, and let $p \in X$ be a point. Let $\nabla$ be a unitary connection on $E$. Show that there exists a unitary frame for $E$ in a neighborhood of $p$, such that the corresponding matrix of connection 1-forms $A$ satisfies $A(p) = 0$. 