1. Let $T: \mathcal{P}_2 \rightarrow \mathcal{P}_3$ be the linear transformation given by

$$T(p(x)) = \frac{dp(x)}{dx} - xp(x),$$

where $\mathcal{P}_2, \mathcal{P}_3$ are the spaces of polynomials of degrees at most 2 and 3 respectively.

(a) Find the matrix representative of $T$ relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$ for $\mathcal{P}_2$ and $\mathcal{P}_3$.

Solution: 

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Find the kernel of $T$.

Solution: The kernel is $\{0\}$.

(c) Find a basis for the range of $T$.

Solution: This is the same as the column space of the matrix in (a), but expressed as elements of $\mathcal{P}_3$. Row reducing the matrix we find that the range has basis $\{-x, 1 - x^2, 2x - x^3\}$.

2. Determine whether the following subsets of $\mathcal{P}_3$ are subspaces.

(a) $U = \{p(x) : p(3) = 0\}$

Solution: This is a subspace. If $p(x), q(x) \in U$, then $(p + q)(3) = p(3) + q(3) = 0$, so $p(x) + q(x) \in U$. Also $(rp)(3) = r \cdot p(3) = 0$, so $rp(x) \in U$ for any $r \in \mathbb{R}$.

(b) $V = \{p(x) : p(0) = 1\}$

Solution: This is not a subspace because it does not contain the zero polynomial.

(c) $W = \{p(x) : \text{the coefficient of } x^2 \text{ in } p(x) \text{ is 0}\}$.

Solution: This is a subspace, because if $p(x), q(x)$ have no $x^2$ term, then neither do $p(x) + q(x)$ and $rq(x)$ for $r \in \mathbb{R}$.

3. Let $M_{m\times n}$ be the vector space of $m \times n$ matrices, with the usual operations of addition and scalar multiplication.

(a) Let $A$ be an $m \times m$ matrix. Is the function

$$T: M_{m\times n} \rightarrow M_{m\times n}$$

given by $T(B) = AB$ a linear transformation?

Solution: It is a linear transformation. We need to check $T(B + C) = T(B) + T(C)$ and $T(rB) = rT(B)$:

$$T(B + C) = A(B + C) = AB + AC = T(B) + T(C)$$

$$T(rB) = A(rB) = rAB = rT(B).$$

(b) Let $V \subset M_{m\times n}$ be the subset consisting of those matrices, whose entries all add up to zero. Is $V$ a subspace of $M_{m\times n}$?

Solution: This is a subspace, since if $A, B$ have entries adding up to zero, then so do $A + B$ and $rA$ for any $r \in \mathbb{R}$. 

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4. Show that the subspaces \( \text{sp}(x - x^2, 2x) \) and \( \text{sp}(x^2, 3x + x^2) \) of \( \mathcal{P}_2 \) are equal.

**Solution:** We first show that \( x - x^2, 2x \in \text{sp}(x^2, 3x + x^2) \):

\[
x - x^2 = \frac{1}{3}(3x + x^2) - \frac{1}{3}x^2 - x^2
\]
\[
2x = \frac{2}{3}(3x + x^2) - x^2.
\]

It follows that \( \text{sp}(x - x^2, 2x) \) is a subspace of \( \text{sp}(x^2, 3x + x^2) \). Similarly, from

\[
x^2 = -(x - x^2) + \frac{1}{2}(2x)
\]
\[
3x + x^2 = -(x - x^2) + \frac{1}{2}(2x) + \frac{3}{2}(2x),
\]

it follows that \( \text{sp}(x^2, 3x + x^2) \) is a subspace of \( \text{sp}(x - x^2, 2x) \). This means that the two subspaces are equal.

5. Find a basis for the subspace \( \text{sp}(1 + x^2, 2x - x^2, 4x + 2) \) of \( \mathcal{P}_3 \).

**Solution:** Using the basis \( \{1, x, x^2, x^3\} \) for \( \mathcal{P}_3 \), we can write the vectors \( 1 + x^2, 2x - x^2, 4x + 2 \) as the columns of the matrix

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 2 & 4 \\
1 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Row reducing the matrix, we find that a basis is given by \( \{1 + x^2, 2x - x^2\} \).

6. Working in the space \( \mathcal{P}_3 \), find the coordinate vector of \( x^2 \), relative to the basis \( \{1, x - 1, (x - 1)^2, (x - 1)^3\} \).

**Solution:** We need to find \( a_1, a_2, a_3, a_4 \) such that

\[
x^2 = a_1 + a_2(x - 1) + a_3(x - 1)^2 + a_4(x - 1)^3.
\]

One can write this as a matrix, and we find \( a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 0 \), so the coordinate vector of \( x^2 \) in the basis is \( [1, 2, 1, 0] \).

7. Compute the determinant

\[
\det \begin{bmatrix}
3 & -2 & 7 & 6 \\
-4 & 0 & 2 & 1 \\
5 & 2 & 0 & -2 \\
2 & 0 & -1 & 0
\end{bmatrix}
\]

**Solution:** Use column and row operations to simplify the calculation:

\[
= -\det \begin{bmatrix}
3 & -2 & 7 & 6 \\
0 & 2 & 0 & -2 \\
2 & 2 & -1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
= -\det \begin{bmatrix}
-2 & 3 & 7 & 6 \\
0 & -4 & 2 & 1 \\
0 & 8 & 7 & 4 \\
0 & 2 & -1 & 0
\end{bmatrix}
\]
\[
= \det \begin{bmatrix}
-2 & 3 & 7 & 6 \\
0 & 2 & -1 & 0 \\
0 & 0 & 11 & 4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
= -44
\]
8. Suppose that $A$ is an $n \times n$ matrix, such that all of the entries of $A$ add up to zero. Is it true that $\det(A) = 0$? 

\textit{Solution:} It is not true, for example 

$$\det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 1.$$