Introduction to Extremal metrics

Preliminary version

Gábor Székelyhidi

Contents

1 Kähler geometry

The goal of this section is to cover some of the background from Kähler geometry that we will need in the course. Rather than formally setting up the theory we will focus on how to do calculations with covariant derivatives and the curvature tensor on Kähler manifolds. For a much more thorough introduction to the subject the standard reference is Griffiths-Harris [22]. Another useful reference with which this course has significant overlap is Tian [52].

1.1 Complex manifolds

A complex manifold M can be thought of as a smooth manifold, on which we have a well defined notion of holomorphic function. More precisely for an integer $n > 0$ (the complex dimension), M is covered by open sets U_α, together with homeomorphisms

$$\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbf{C}^n,$$

such that the "transition maps" $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic wherever they are defined. A function $f : M \to \mathbf{C}$ is then holomorphic, if the composition $f \circ \varphi_\alpha^{-1}$ is holomorphic on V_α, for all α. Using these charts, near any point $p \in M$ there exists a holomorphic coordinate system $z^1, \ldots, z^n$, consisting of complex valued functions with $z^i(p) = 0$ for each i. Moreover if $w^1, \ldots, w^n$ form a different holomorphic coordinate system, then each w^i is a holomorphic function of the $z^1, \ldots, z^n$.

Example 1.1 (The Riemann sphere)**.** We let $M = S^2$, and we think of $S^2 \subset \mathbf{R}^3$ as the unit sphere. Identify the xy-plane in $\mathbf{R}^3$ with $\mathbf{C}$. We define two charts. Let U_1 be the complement of the "north pole", i.e. $U_1 = S^2 \setminus \{(0,0,1)\}$, and define

$$\varphi : U_1 \to \mathbf{C}$$

to be stereographic projection from the north pole to the xy-plane. Similarly let $U_2 = S^2 \setminus \{(0,0,-1)\}$ be the complement of the south pole, and let

$$\psi : U_2 \to \mathbf{C}$$

be the composition of stereographic projection to the xy-plane from the south pole, with complex conjugation. One can then compute that

$$\psi \circ \varphi^{-1}(z) = \frac{1}{z} \text{ for } z \in \mathbf{C} \setminus \{0\}.$$

Since this transition function is holomorphic, our two charts give S^2 the structure of a complex manifold. Note that if we do not compose the projection with complex conjugation when defining ψ, then even the orientations defined by φ and ψ would not match, although the two charts would still give S^2 the structure of a smooth manifold.

Example 1.2 (Complex projective space)**.** The complex projective space $\mathbf{CP}^n$ is defined to be the space of complex lines in $\mathbf{C}^{n+1}$. In other words points of $\mathbf{CP}^n$ are $(n+1)$-tuples $[Z_0 : \ldots : Z_n]$, where not every entry is zero, and we identify

$$[Z_0 : \ldots : Z_n] = [\lambda Z_0 : \ldots : \lambda Z_n]$$

for all $\lambda \in \mathbf{C} \setminus \{0\}$. As a topological space $\mathbf{CP}^n$ inherits the quotient topology from $\mathbf{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ under this equivalence relation. We call the $Z_0, \ldots, Z_n$ homogeneous coordinates. To define the complex structure we will use $n+1$ charts. For $i \in \{0, 1, \ldots, n\}$, let

$$U_i = \left\{ [Z_0 : \ldots : Z_n] \,\middle|\, Z_i \neq 0 \right\},$$

and

$$\begin{aligned} \varphi_i : U_i &\to \mathbf{C}^n \\ [Z_0 : \ldots : Z_n] &\mapsto \left(\frac{Z_0}{Z_i}, \ldots, \widehat{\frac{Z_i}{Z_i}}, \ldots, \frac{Z_n}{Z_i} \right), \end{aligned}$$

where the $\dfrac{Z_i}{Z_i}$ term is omitted. It is then easy to check that the transition functions are holomorphic. For example using coordinates $w^1, \ldots, w^n$ on $\mathbf{C}^n$ we have

$$\varphi_1 \circ \varphi_0^{-1}(w^1, \ldots, w^n) = \left(\frac{1}{w^1}, \frac{w^2}{w^1}, \ldots, \frac{w^n}{w^1} \right). \tag{1}$$

In the case $n = 1$ we obtain two charts with the same transition function as in the previous example, so $\mathbf{CP}^1 = S^2$ as complex manifolds.

Topologically $\mathbf{CP}^n$ can be seen as a quotient S^{2n+1}/S^1, where $S^{2n+1} \subset \mathbf{C}^{n+1}$ is the unit sphere, and S^1 acts as multiplication by unit length complex numbers. It follows that $\mathbf{CP}^n$ is compact.

Example 1.3 (Projective manifolds)**.** Suppose that $f_1, \ldots, f_k$ are homogeneous polynomials in $Z_0, \ldots, Z_n$. Even though the f_i are not well-defined functions on $\mathbf{CP}^n$ (we will later see that they are sections of line bundles), their zero sets are well-defined. Let $V \subset \mathbf{CP}^n$ be their common zero set

$$V = \left\{ [Z_0 : \ldots : Z_n] \,\middle|\, f_i(Z_0, \ldots, Z_n) = 0 \text{ for } i = 1, \ldots, k \right\}.$$

If V is a smooth submanifold, then it is a complex manifold and charts can be constructed using the implicit function theorem. Being closed subsets of a compact space, projective manifolds are compact.

These projective manifolds are general enough that in this course they are essentially the only complex manifolds with which we will be concerned. They lie at the intersection of complex differential geometry and algebraic geometry and we will require tools from both fields. In particular the basic question we will ask is differential geometric in nature, about the existence of certain special metrics on projective manifolds. In studying this question, however, one is naturally led to consider the behaviour of projective manifolds in families, and their degenerations to possibly singular limiting spaces. Algebraic geometry will provide a powerful tool to study such problems.

1.2 Almost complex structures

An alternative way to introduce complex manifolds is through almost complex structures.

Definition 1.4. An almost complex structure on an smooth manifold M is an endomorphism $J : TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{Id}$, where Id is the identity map.

In other words an almost complex structure equips the tangent space at each point with a linear map which behaves like multiplication by $\sqrt{-1}$. The dimension of M must then be even, since any endomorphism of an odd dimensional vector space has a real eigenvalue, which could not square to -1.

Example 1.5. If M is a complex manifold, then the holomorphic charts identify each tangent space T_pM with $\mathbf{C}^n$, so we can define $J(v) = \sqrt{-1}v$ for $v \in T_pM$, giving an almost complex structure. The fact that the transition functions are holomorphic means precisely that multiplication by $\sqrt{-1}$ is compatible under the different identifications of T_pM with $\mathbf{C}^n$ using different charts.

If $z^1, \ldots, z^n$ are holomorphic coordinates and $z^i = x^i + \sqrt{-1}y^i$ for real functions x^i, y^i, then we can also write

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}.$$

Definition 1.6. An almost complex structure is called integrable, if it arises from holomorphic charts as in the previous example. We will use the term "complex structure" to mean an integrable almost complex structure.

On complex manifolds it is convenient to work with the complexified tangent bundle

$$T^{\mathbf{C}}M = TM \otimes_{\mathbf{R}} \mathbf{C}.$$

In terms of local holomorphic coordinates it is convenient to use the basis

$$\left\{ \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \overline{z}^1}, \dots, \frac{\partial}{\partial \overline{z}^n} \right\}, \tag{2}$$

where in terms of the real and imaginary parts $z^i = x^i + \sqrt{-1}y^i$ we have

$$\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i} \right), \text{ and } \frac{\partial}{\partial \overline{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i} \right). \tag{3}$$

The endomorphism J extends to a complex linear endomorphism of $T^{\mathbf{C}}M$, and induces a decomposition of this bundle pointwise into the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces

$$T^{\mathbf{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

In terms of local holomorphic coordinates $T^{1,0}M$ is spanned by the $\dfrac{\partial}{\partial z^i}$ while $T^{0,1}M$ is spanned by the $\dfrac{\partial}{\partial \overline{z}^i}$.

Similarly we can complexify the cotangent bundle to obtain $\Omega^1_{\mathbf{C}}M$, which is decomposed according to the eigenvalues of the endomorphism dual to J (which we will still denote by J) into

$$\Omega^1_{\mathbf{C}}M = \Omega^{1,0}M \oplus \Omega^{0,1}M.$$

In terms of coordinates, $\Omega^{1,0}$ is spanned by $dz^1, \dots, dz^n$, while $\Omega^{0,1}$ is spanned by $d\overline{z}^1, \dots, d\overline{z}^n$, where

$$dz^i = dx^i + \sqrt{-1}dy^i, \text{ and } d\overline{z}^i = dx^i - \sqrt{-1}dy^i.$$

Moreover $\{dz^1, \dots, dz^n, d\overline{z}^1, \dots, d\overline{z}^n\}$ is the basis dual to (2).

The decomposition extends to higher degree forms

$$\Omega^r_{\mathbf{C}}M = \bigoplus_{p+q=r} \Omega^{p,q}M,$$

where $\Omega^{p,q}M$ is locally spanned by

$$dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \dots \wedge d\overline{z}^{j_q}.$$

On a complex manifold the decomposition of forms gives rise to a decomposition of the exterior derivative as $d = \partial + \overline{\partial}$, where

$$\partial : \Omega^{p,q}M \to \Omega^{p+1,q}M$$
$$\overline{\partial} : \Omega^{p,q}M \to \Omega^{p,q+1}M$$

are two projections of d. A useful observation is that $\overline{\partial}\alpha = \overline{\partial\overline{\alpha}}$ for any form α.

Example 1.7. A function $f : M \to \mathbf{C}$ is holomorphic if and only if $\overline{\partial} f = 0$, since

$$\overline{\partial} f = \frac{\partial f}{\partial \overline{z}^1} d\overline{z}^1 + \ldots + \frac{\partial f}{\partial \overline{z}^n} d\overline{z}^n,$$

and $\dfrac{\partial f}{\partial \overline{z}^i}$ are the Cauchy-Riemann equations.

Example 1.8. For a function $f : M \to \mathbf{R}$, the form $\sqrt{-1}\partial\overline{\partial} f$ is a real $(1,1)$-form, a kind of complex Hessian of f. In particular if $f : \mathbf{C} \to \mathbf{R}$, then

$$\begin{aligned}
\sqrt{-1}\partial\overline{\partial} f &= \sqrt{-1}\left(\frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}}\right) dz \wedge d\overline{z} \\
&= \frac{\sqrt{-1}}{4}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + \sqrt{-1}\frac{\partial f}{\partial y}\right)(dx + \sqrt{-1}dy) \wedge (dx - \sqrt{-1}dy) \\
&= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy.
\end{aligned}$$

1.3 Hermitian and Kähler metrics

Let M be a complex manifold with complex structure J. We will be interested in Riemannian metrics on M which are compatible with the complex structure in a particularly nice way. Recall that a Riemannian metric is a positive definite symmetric bilinear form on each tangent space.

Definition 1.9. A Riemannian metric g on M is Hermitian if $g(JX, JY) = g(X, Y)$ for any tangent vectors X, Y. In other words we require J to be an orthogonal transformation on each tangent space.

Given a Hermitian metric g we define $\omega(X, Y) = g(JX, Y)$ for any X, Y. Then ω is anti-symmetric in X, Y and one can check that in this way ω defines a real 2-form of type $(1,1)$.

Definition 1.10. A Hermitian metric g is Kähler, if the associated 2-form ω is closed, i.e. $d\omega = 0$. Then ω is called the Kähler form, but often we will call ω the Kähler metric and make no mention of g.

In local coordinates $z^1, \ldots, z^n$ a Hermitian metric is determined by the components $g_{j\bar{k}}$ where

$$g_{j\bar{k}} = g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}\right),$$

and we are extending g to complex tangent vectors by complex linearity in both entries. The Hermitian condition implies that for any j, k we have

$$g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) = g\left(\frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}\right) = 0.$$

In terms of the components $g_{j\bar{k}}$ we can therefore write

$$g = \sum_{j,k} g_{j\bar{k}}(dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j).$$

Note that the bar on $\bar{k}$ in the components $g_{j\bar{k}}$ is used to remember the distinction between holomorphic and anti-holomorphic components.

The symmetry of g implies that $\overline{g_{j\bar{k}}} = g_{k\bar{j}}$, and the positivity of g means that $g_{j\bar{k}}$ is a positive definite Hermitian matrix at each point. The associated 2-form ω can be written as

$$\omega = \sqrt{-1}\sum_{j,k} g_{j\bar{k}} dz^j \wedge d\bar{z}^k,$$

and finally g is Kähler if for all i, j, k we have

$$\frac{\partial}{\partial z^i} g_{j\bar{k}} = \frac{\partial}{\partial z^j} g_{i\bar{k}}.$$

Example 1.11 (Fubini-Study metric)**.** The complex projective space $\mathbf{CP}^n$ has a natural Kähler metric ω_{FS} called the Fubini-Study metric. To construct it, recall the projection map $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$. A section s over an open set $U \subset \mathbf{CP}^n$ is a holomorphic map $s : U \to \mathbf{C}^{n+1}$ such that $\pi \circ s$ is the identity. Given such a section we define

$$\omega_{FS} = \sqrt{-1}\partial\overline{\partial} \log \|s\|^2.$$

To check that this is well-defined, note that if s' is another section over an open set V, then on the intersection $U \cap V$ we have $s' = fs$ for a holomorphic function $f : U \cap V \to \mathbf{C} \setminus \{0\}$, and

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial} \log \|fs\|^2 &= \sqrt{-1}\partial\overline{\partial} \log \|s\|^2 + \sqrt{-1}\partial\overline{\partial} \log f + \sqrt{-1}\partial\overline{\partial} \log \overline{f} \\ &= \sqrt{-1}\partial\overline{\partial} \log \|s\|^2.\end{aligned}$$

Since sections exist over small open sets U, we obtain a well defined, closed $(1,1)$-form on $\mathbf{CP}^n$. The form ω_{FS} is $U(n+1)$-invariant, and $U(n+1)$ acts transitively on $\mathbf{CP}^n$ so it is enough to check that the corresponding Hermitian matrix is positive definite at a single point. At the point $[1:0:\ldots:0]$ let us use local holomorphic coordinates

$$z^i = \frac{Z_i}{Z_0} \text{ for } i = 1, \ldots, n,$$

on the chart U_0. A section is then given by

$$s(z^1, \ldots, z^n) = (1, z^1, \ldots, z^n),$$

so

$$\omega_{FS} = \sqrt{-1}\partial\overline{\partial}\log(1 + |z^1|^2 + \ldots |z^n|^2). \tag{4}$$

At the origin this equals $\sqrt{-1}\sum_i dz^i \wedge d\bar{z}^i$. The corresponding Hermitian matrix is the identity, which is positive definite. From (4) it is clear that ω_{FS} is closed since it is locally exact.

Example 1.12. If $V \subset \mathbf{CP}^n$ is a projective manifold, then ω_{FS} restricted to V gives a Kähler metric on V, since the exterior derivative commutes with pulling back differential forms.

Since the Kähler form ω is a closed real form, it defines a cohomology class $[\omega]$ in $H^2(M, \mathbf{R})$. A fundamental result is the $\partial\overline{\partial}$-lemma, which shows that on a compact manifold, Kähler metrics in a fixed cohomology class can be parametrized by real valued functions.

Lemma 1.13 ($\partial\overline{\partial}$-lemma)**.** *Let M be a compact Kähler manifold. If ω and η are two real (1,1)-forms in the same cohomology class, then there is a function $f : M \to \mathbf{R}$ such that*

$$\eta = \omega + \sqrt{-1}\partial\overline{\partial}f.$$

Proof. The proof of this result requires some ideas from Hodge theory on Kähler manifolds, which we have not discussed. Because of the fundamental nature of the result we give the proof in any case.

Let g be a Kähler metric on M. Since $[\eta] = [\omega]$ and η, ω are real forms, there exists a real 1-form α such that

$$\eta = \omega + d\alpha.$$

Let us decompose $\alpha = \alpha^{1,0} + \alpha^{0,1}$ into its $(1,0)$ and $(0,1)$ parts, where $\alpha^{0,1} = \overline{\alpha^{1,0}}$ since α is real. Since η, ω are (1,1)-forms, we have

$$\eta = \omega + \overline{\partial}\alpha^{1,0} + \partial\alpha^{0,1}, \tag{5}$$

and $\partial\alpha^{1,0} = \overline{\partial}\alpha^{0,1} = 0$. The function $\partial^*\alpha^{1,0}$ defined by

$$\partial^*\alpha^{1,0} = -g^{j\bar{k}}\nabla_{\bar{k}}\alpha_j$$

has zero integral on M, so using Theorem 2.11 in Section 2.4 there is a function f such that

$$\partial^*\alpha^{1,0} = \Delta f = -\partial^*\partial f.$$

Then

$$\partial(\alpha^{1,0} + \partial f) = 0, \qquad \text{and } \partial^*(\alpha^{1,0} + \partial f) = 0,$$

so $\alpha^{1,0} + \partial f$ is a ∂-harmonic form. Since g is Kähler, the form is also $\overline{\partial}$-harmonic, so in particular it is $\overline{\partial}$-closed (see Exercise 1.1), so

$$\overline{\partial}\alpha^{1,0} = -\overline{\partial}\partial f.$$

From (5) we then have

$$\eta - \omega = -\overline{\partial}\partial f - \partial\overline{\partial f} = \partial\overline{\partial}(f - \overline{f}) = \sqrt{-1}\partial\overline{\partial}\mathrm{Im}(f),$$

where $\mathrm{Im}(f)$ is the imaginary part of f. □

The next result shows that if we have a Kähler metric, then we can choose particularly nice holomorphic coordinates near any point. This will be very useful in computations later on.

Proposition 1.14 (Normal coordinates)**.** *If g is a Kähler metric, then around any point $p \in M$ we can choose holomorphic coordinates $z^1, \ldots, z^n$ such that the components of g at the point p satisfy*

$$g_{j\bar{k}}(p) = \delta_{jk} \ \text{ and } \frac{\partial}{\partial z^i}g_{j\bar{k}}(p) = \frac{\partial}{\partial \bar{z}^i}g_{j\bar{k}}(p) = 0, \tag{6}$$

where δ_{jk} is the identity matrix, i.e. $\delta_{jk} = 0$ if $j \neq k$, and $\delta_{jk} = 1$ if $j = k$.

Proof. It is equivalent to (6) to require that the Kähler form satisfies

$$\omega = \sqrt{-1}\sum_{j,k}\left(\delta_{jk} + O(|z|^2)\right)dz^j \wedge d\bar{z}^k, \tag{7}$$

where $O(|z|^2)$ denotes terms which are at least quadratic in the $z^i, \bar{z}^i$.

First we choose coordinates w^i such that

$$\omega = \sqrt{-1}\sum_{j,k}\left(\delta_{jk} + \sum_l (a_{j\bar{k}l}w^l + a_{j\bar{k}\bar{l}}\bar{w}^l) + O(|w|^2)\right) dw^j \wedge d\bar{w}^k. \tag{8}$$

Next we define new coordinates z^i which satisfy

$$w^i = z^i - \frac{1}{2}\sum_{j,k} b_{ijk}z^j z^k,$$

for some coefficients b_{ijk}, such that $b_{ijk} = b_{ikj}$. Then

$$dw^i = dz^i - \sum_{j,k} b_{ijk}z^j dz^k,$$

so we can compute

$$\omega = \sqrt{-1}\sum_{j,k}\Big(\delta_{jk} + \sum_l (a_{j\bar{k}l}z^l + a_{j\bar{k}\bar{l}}\bar{z}^l - b_{klj}z^l - \overline{b_{jlk}}\bar{z}^l) + O(|z|^2)\Big)dz^j \wedge d\bar{z}^k.$$

If ω is Kähler, then from (8) we know that $a_{j\bar{k}l} = a_{l\bar{k}j}$, so we can define $b_{klj} = a_{j\bar{k}l}$. Then we have

$$a_{j\bar{k}\bar{l}} = \overline{a_{k\bar{j}l}} = \overline{b_{jlk}},$$

so all the linear terms cancel. □

In Riemannian geometry we can always choose normal coordinates in which the first derivatives of the metric vanish at a given point, and of course this result applies to any Hermitian metric too. The point of the previous result is that if the metric is Kähler, then we can even find a *holomorphic* coordinate system in which the first derivatives of the metric vanish at a point. Conversely it is clear from the expression (7) that if such holomorphic normal coordinates exist, then $d\omega = 0$, so the metric is Kähler.

1.4 Covariant derivatives and curvature

Given a Kähler manifold (M, ω) we use the Levi-Civita connection ∇ to differentiate tensor fields. This satisfies $\nabla g = \nabla\omega = \nabla J = 0$. In terms of local holomorphic coordinates $z^1, \ldots, z^n$, we will use the following notation for the different derivatives:

$$\nabla_i = \nabla_{\partial/\partial z^i}, \quad \nabla_{\bar{i}} = \nabla_{\partial/\partial \bar{z}^i}, \quad \partial_i = \frac{\partial}{\partial z^i}, \quad \partial_{\bar{i}} = \frac{\partial}{\partial \bar{z}^i}.$$

The connection is determined by the Christoffel symbols Γ^i_{jk}, given by

$$\nabla_j \frac{\partial}{\partial z^k} = \sum_i \Gamma^i_{jk} \frac{\partial}{\partial z^i},$$

while at the same time the Kähler condition can be used to show that

$$\nabla_{\bar{i}} \frac{\partial}{\partial z^k} = 0.$$

The Levi-Civita connection is symmetric (torsion free) so we have $\Gamma^i_{jk} = \Gamma^i_{kj}$. In addition $\nabla_{\bar{i}} T = \overline{\nabla_i \overline{T}}$ for any tensor T. Covariant derivatives of tensor fields can then be computed using the product rule for derivatives, remembering that on functions the covariant derivatives coincide with the usual partial derivatives.

Example 1.15. To find the covariant derivatives of the form dz^k, we differentiate the relation

$$dz^k \left(\frac{\partial}{\partial z^j} \right) = \delta^k_j,$$

where δ^k_j is the identity matrix. We get

$$(\nabla_i dz^k) \frac{\partial}{\partial z^j} + dz^k \left(\nabla_i \frac{\partial}{\partial z^j} \right) = 0,$$

from which we can calculate that

$$(\nabla_i dz^k) \frac{\partial}{\partial z^j} = -\Gamma^k_{ij},$$

and similarly $(\nabla_i dz^k) \dfrac{\partial}{\partial \bar{z}^j} = 0$. It follows that

$$\nabla_i dz^k = -\sum_j \Gamma^k_{ij} dz^j.$$

From now we will start using summation convention which means that we sum over repeated indices. If we are consistent, then each repeated index should appear once on top and once on the bottom. Usually we will write a tensor such as $a_{i\bar{j}} dz^i \otimes d\bar{z}^j$ (summing over i, j) as just $a_{i\bar{j}}$. Note however that Γ^i_{jk} is not a tensor since it does not transform in the right way under changes of coordinates.

Example 1.16. We compute covariant derivatives of a tensor $a_{i\bar{j}}dz^i \otimes d\bar{z}^j$ using the product rule, namely

$$\begin{aligned}\nabla_{\bar{p}}(a_{i\bar{j}}dz^i \otimes d\bar{z}^j) =& (\partial_{\bar{p}}a_{i\bar{j}})dz^i \otimes d\bar{z}^j + a_{i\bar{j}}(\nabla_{\bar{p}}dz^i) \otimes d\bar{z}^j + a_{i\bar{j}}dz^i \otimes (\nabla_{\bar{p}}d\bar{z}^j) \\ =& (\partial_{\bar{p}}a_{i\bar{j}})dz^i \otimes d\bar{z}^j - a_{i\bar{j}}dz^i \otimes (\Gamma^{\bar{j}}_{\bar{p}\bar{l}}d\bar{z}^l) \\ =& \left(\partial_{\bar{p}}a_{i\bar{j}} - \Gamma^{\bar{l}}_{\bar{p}\bar{j}}a_{i\bar{l}}\right) dz^i \otimes d\bar{z}^j.\end{aligned}$$

We can write this formula more concisely as

$$\nabla_{\bar{p}}a_{i\bar{j}} = \partial_{\bar{p}}a_{i\bar{j}} - \Gamma^{\bar{l}}_{\bar{p}\bar{j}}a_{i\bar{l}},$$

and similar formulas for more general tensors can readily be derived.

Lemma 1.17. *In terms of the metric $g_{j\bar{k}}$ the Christoffel symbols are given by*

$$\Gamma^i_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}},$$

where $g^{i\bar{l}}$ is the matrix inverse to $g_{i\bar{l}}$.

Proof. The Levi-Civita connection satisfies $\nabla g = 0$. In coordinates this means

$$0 = \nabla_j g_{k\bar{l}} = \partial_j g_{k\bar{l}} - \Gamma^p_{jk}g_{p\bar{l}},$$

so

$$g^{i\bar{l}}\partial_j g_{k\bar{l}} = \Gamma^p_{jk}g_{p\bar{l}}g^{i\bar{l}} = \Gamma^p_{jk}\delta^i_p = \Gamma^i_{jk}.$$

□

Covariant derivatives do not commute in general, and the failure to commute is measured by the curvature. The curvature is a 4-tensor $R_{i\ k\bar{l}}^{\ j}$, where we will often raise or lower indices using the metric, for example $R_{i\bar{j}k\bar{l}} = g_{p\bar{j}}R_{i\ k\bar{l}}^{\ p}$ (note that the position of the indices is important). The curvature is defined by

$$(\nabla_k\nabla_{\bar{l}} - \nabla_{\bar{l}}\nabla_k)\frac{\partial}{\partial z^i} = R_{i\ k\bar{l}}^{\ j}\frac{\partial}{\partial z^j},$$

while ∇_k commutes with ∇_l, and $\nabla_{\bar{k}}$ commutes with $\nabla_{\bar{l}}$. In terms of the Christoffel symbols we can compute

$$R_{i\ k\bar{l}}^{\ j} = -\partial_{\bar{l}}\Gamma^j_{ki},$$

from which we find that in terms of the metric

$$R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}g_{i\bar{j}} + g^{p\bar{q}}(\partial_k g_{i\bar{q}})(\partial_{\bar{l}}g_{p\bar{j}}).$$

In terms of normal coordinates around a point p we have $R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}g_{i\bar{j}}$ at p. In other words the curvature tensor of a Kähler metric is the obstruction to finding holomorphic coordinates in which the metric agrees with the Euclidean metric up to 2nd order. It turns out that if we write out the Taylor expansion of the metric in normal coordinates, then each coefficient will only depend on covariant derivatives of the curvature. In particular if the curvature vanishes in a neighborhood of a point, then in normal coordinates the metric is just given by the Euclidean metric.

The curvature satisfies various identities, see Exercise 1.4. The Ricci curvature is defined to be the contraction

$$R_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}},$$

and the scalar curvature is

$$R = g^{i\bar{j}}R_{i\bar{j}}.$$

Lemma 1.18. *In local coordinates*

$$R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det(g_{p\bar{q}}).$$

Proof. Using the formulas above, we have

$$\begin{aligned}
-\partial_{\bar{j}}\partial_i\log\det(g_{p\bar{q}}) &= -\partial_{\bar{j}}(g^{p\bar{q}}\partial_i g_{p\bar{q}}) \\
&= -\partial_{\bar{j}}\Gamma^p_{ip} \\
&= R_p{}^p{}_{i\bar{j}} \\
&= R_{i\bar{j}}.
\end{aligned}$$

□

As a consequence the Ricci form $\mathrm{Ric}(\omega)$ defined by

$$\mathrm{Ric}(\omega) = \sqrt{-1}R_{i\bar{j}}dz^i\wedge d\bar{z}^j = -\sqrt{-1}\partial\overline{\partial}\log\det(g)$$

in local coordinates is a closed real $(1,1)$-form. Moreover if h is another Kähler metric on M, then $\dfrac{\det(h)}{\det(g)}$ is a globally defined function, so the difference of Ricci forms

$$\mathrm{Ric}(h) - \mathrm{Ric}(g) = -i\partial\overline{\partial}\log\frac{\det(h)}{\det(g)}$$

is an exact form. The cohomology class $[\mathrm{Ric}(g)]$ is therefore independent of the choice of Kähler metric. The first Chern class of M is defined to be the cohomology class

$$c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}(g)] \in H^2(M, \mathbf{R}).$$

It turns out that with this normalization $c_1(M)$ is actually an integral cohomology class.

The fundamental result about the Ricci curvature of Kähler manifolds is Yau's solution of the Calabi conjecture.

Theorem 1.19 (Calabi-Yau theorem)**.** *Let (M, ω) be a compact Kähler manifold, and let α be a real (1,1)-form representing $c_1(M)$. Then there exists a unique Kähler metric η on M with $[\eta] = [\omega]$, such that* $\mathrm{Ric}(\eta) = 2\pi\alpha$.

In particular if $c_1(M) = 0$, then every Kähler class contains a unique Ricci flat metric. This provides our first example of a canonical Kähler metric, and it is a very special instance of an extremal metric. We will discuss the proof of this theorem in Section 3.4.

1.5 Vector bundles

A holomorphic vector bundle E over a complex manifold M is a holomorphic family of complex vector spaces parametrized by M. E is itself a complex manifold, together with a holomorphic projection $\pi : E \to M$, and the family is locally trivial so M has an open cover $\{U_\alpha\}$ such that we have biholomorphisms (trivializations)

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}^r, \tag{9}$$

for some integer $r > 0$ called the rank of E. Under the trivialization φ_α, π corresponds to projection onto U_α. The trivializations are related by holomorphic transition maps

$$\begin{aligned} \varphi_\beta \circ \varphi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbf{C}^r &\to (U_\alpha \cap U_\beta) \times \mathbf{C}^r \\ (p, v) &\mapsto (p, \varphi_{\alpha\beta}(p)v), \end{aligned} \tag{10}$$

which at each point $p \in U_\alpha \cap U_\beta$ gives a linear isomorphism $\varphi_{\alpha\beta}(p)$ from $\mathbf{C}$ to $\mathbf{C}$. The matrix valued functions $\varphi_{\alpha\beta}$ satisfy the compatibility (or cocycle) condition

$$\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \varphi_{\alpha\gamma}. \tag{11}$$

Conversely any set of holomorphic matrix valued functions $\varphi_{\alpha\beta}$ satisfying the cocycle conditions defines a vector bundle.

A holomorphic section of a vector bundle E is a holomorphic map $s : M \to E$ such that $\pi \circ s$ is the identity map. A local trivialization φ_α as in (9) gives rise to local holomorphic sections corresponding to constant functions on U_α. In particular a basis of $\mathbf{C}^r$ gives rise to local holomorphic sections $\mathbf{s}_1, \ldots, \mathbf{s}_r$ which we call a local holomorphic frame. The values of the $\mathbf{s}_i$ span the fiber $E_p = \pi^{-1}(p)$ at each point $p \in U_\alpha$. All other local holomorphic sections over U_α can be written as

$$f = \sum_{i=1}^{r} f^i \mathbf{s}_i,$$

where each f^i is a holomorphic function on U_α. We write the space of global holomorphic sections as $H^0(M, E)$, since this forms the first term in a sequence of cohomology spaces $H^i(M, E)$. Although they are fundamental objects, we will not be using these spaces for $i > 0$ in this course. An important property which we will discuss later is that $H^0(M, E)$ is finite dimensional if M is compact.

Example 1.20. The $(1, 0)$ part of the cotangent bundle $\Omega^{1,0}M$ is a rank n holomorphic vector bundle over M, where $\dim_{\mathbf{C}} M = n$. In a local chart with holomorphic coordinates $z^1, \ldots, z^n$ a trivialization is given by the holomorphic frame $dz^1, \ldots, dz^n$. The transition map to a different chart is determined by the Jacobian matrix of the coordinate transformation. This bundle is the holomorphic cotangent bundle.

Natural operations on vector spaces can be extended to vector bundles, such as taking tensor products, direct sums, duals, etc.

Example 1.21. On a complex manifold of dimension n we can form the n-th exterior power of the holomorphic cotangent bundle. This is a line bundle (rank 1 vector bundle) denoted by K_M and is called the canonical bundle of M:

$$K_M = \bigwedge^n \Omega^{1,0} M = \Omega^{n,0} M.$$

In local holomorphic coordinates a frame is given by $dz^1 \wedge \ldots \wedge dz^n$, and the transition functions are given by Jacobian determinants.

Example 1.22 (Line bundles over $\mathbf{CP}^n$)**.** Since $\mathbf{CP}^n$ is the space of complex lines in $\mathbf{C}^{n+1}$ we can construct a line bundle denoted by $\mathcal{O}(-1)$ over $\mathbf{CP}^n$

by assigning to each point the line it parametrizes. A natural way to think of $\mathcal{O}(-1)$ is as a subbundle of the trivial bundle $\mathbf{CP}^n \times \mathbf{C}^{n+1}$. Recall the charts U_i from Example 1.2. It is a good exercise to work out that under suitable trivializations the transition functions correponding to these charts are given by

$$\varphi_{jk}([Z_0 : \ldots : Z_n]) = \frac{Z_k}{Z_j}, \tag{12}$$

in terms of homogeneous coordinates. Note that while Z_j, Z_k are not well-defined functions on $U_j \cap U_k$, their quotient is well-defined.

Since $\mathcal{O}(-1)$ is a subbundle of the trivial bundle, any global holomorphic section of $\mathcal{O}(-1)$ gives rise to a holomorphic map $s : \mathbf{CP}^n \to \mathbf{C}^{n+1}$. The components of s are holomorphic functions on a compact complex manifold, so they are constant. Therefore s itself is a constant map. It is easy to check that non-zero constant maps do not give rise to sections of $\mathcal{O}(-1)$, so $H^0(\mathbf{CP}^n, \mathcal{O}(-1)) = \{0\}$.

The dual of $\mathcal{O}(-1)$ is denoted by $\mathcal{O}(1)$, and by taking tensor powers we obtain line bundles $\mathcal{O}(l)$ for all integers l. The transition functions $\varphi_{jk}^{(l)}$ of $\mathcal{O}(l)$ are given similarly to (12) by

$$\varphi_{jk}^{(l)}([Z_0 : \ldots : Z_n]) = \left(\frac{Z_j}{Z_k}\right)^l, \tag{13}$$

and the global sections of $\mathcal{O}(l)$ for $l \geqslant 0$ can be thought of as homogeneous polynomials in $Z_0, \ldots, Z_n$ of degree l. In terms of local trivializations, if f is a homogeneous polynomial of degree l, then over the chart U_j we have a holomorphic function $Z_j^{-l} f$. Over different charts these functions patch up using the transition functions (13), so they give rise to a global section of $\mathcal{O}(l)$. It turns out that on $\mathbf{CP}^n$ every line bundle is given by $\mathcal{O}(l)$ for some $l \in \mathbf{Z}$.

1.6 Connections and curvature of line bundles

The Levi-Civita connection that we used before is a canonical connection on the tangent bundle of a Riemannian manifold. Analogously there is a canonical connection on an arbitrary holomorphic vector bundle equipped with a Hermitian metric, called the Chern connection.

A Hermitian metric h on a complex vector bundle is a smooth family of Hermitian inner products on the fibers. In other words, for any two local sections (not necessarily holomorphic) s_1, s_2 we obtain a function $\langle s_1, s_2 \rangle_h$, which satisfies $\langle s_2, s_1 \rangle_h = \overline{\langle s_1, s_2 \rangle_h}$. We can think of the inner product as a section of $E^* \otimes \overline{E^*}$. The Chern connection on a holomorphic vector bundle is then the unique connection on E such that the derivative of the inner product is zero, and $\nabla_{\bar{i}} s = 0$ for any local holomorphic section of E. The derivative of the inner product h is zero if and only if

$$\partial_k(\langle s_1, s_2\rangle_h) = \langle \nabla_k s_1, s_2\rangle_h + \langle s_1, \nabla_{\bar{k}} s_2\rangle_h,$$

and a similar formula holds for $\partial_{\bar{k}}$. Just as before, covariant derivatives do not commute in general, and the curvature $F_{k\bar{l}}$ is defined by

$$F_{k\bar{l}} = \nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k,$$

where both sides are endomorphisms of E for each k, l.

Remark 1.23. Note that on a complex manifold M with a Hermitian metric, the holomorphic tangent bundle $T^{1,0}M$ has two natural connections. There is the Levi-Civita connection that we were concerned with in Section 1.4, and also the Chern connection. It turns out that these two connections coincide if and only if the metric is Kähler (Exercise 1.5). In this case the curvature tensor $R_{i\ k\bar{l}}^{\ j}$ we defined earlier is the same as the curvature $F_{k\bar{l}}$ of the Chern connection on $T^{1,0}$ except in the latter the endomorphism part is suppressed in the notation.

Let us focus now on the case of line bundles, since in this course we will mainly be concerned with those. On a line bundle a Hermitian metric at any point is determined by the norm of any given non-vanishing section at that point. Let s be a local non-vanishing holomorphic section of L, and write

$$h(s) = \langle s, s\rangle_h.$$

Then locally any other section of L can be written as fs for some function f, and the norm of fs is $|fs|_h^2 = |f|^2 h(s)$. In particular we have functions A_k (analogous to the Christoffel symbols before), defined by

$$\nabla_k s = A_k s.$$

Then the curvature is determined by (remembering that s is holomorphic)

$$F_{k\bar{l}} s = -\nabla_{\bar{l}} \nabla_k s = -\nabla_{\bar{l}}(A_k s) = -(\partial_{\bar{l}} A_k) s,$$

so $F_{k\bar{l}} = -\partial_{\bar{l}} A_k$. To determine A_k we use the defining properties of the Chern connection, to get

$$\partial_k h(s) = \langle \nabla_k s, s\rangle_h = A_k h(s),$$

so $A_k = h(s)^{-1}\partial_k h(s)$. It follows that

$$F_{k\bar{l}} = -\partial_{\bar{l}}(h(s)^{-1}\partial_k h(s)) = -\partial_{\bar{l}}\partial_k \log h(s).$$

We can summarize these calculations as follows.

Lemma 1.24. *The curvature of the Chern connection of a holomorphic line bundle equipped with a Hermitian metric is given by*

$$F_{k\bar{l}} = -\partial_k\partial_{\bar{l}} \log h(s),$$

where $h(s) = \langle s, s\rangle_h$ for a local holomorphic section s.

Note the similarity with Lemma 1.18 dealing with the Ricci curvature. The relationship between the two results is that as we remarked above, the Levi-Civita connection of a Kähler metric coincides with the Chern connection on its holomorphic tangent bundle. The determinant of the metric defines a Hermitian metric on the top exterior power $\bigwedge T^{1,0}$, and the Ricci curvature is the curvature of the induced connection on this line bundle.

Just as in the case of the Ricci curvature, Lemma 1.24 implies that the form locally defined by

$$F(h) = \sqrt{-1}F_{k\bar{l}}dz^k \wedge d\bar{z}^l = -\sqrt{-1}\partial\bar{\partial}\log h(s)$$

is a closed real $(1,1)$-form. Any other Hermitian metric can be written as $e^{-f}h$ for a globally defined function f, and we can check that

$$F(e^{-f}h) - F(h) = \sqrt{-1}\partial\bar{\partial}f, \tag{14}$$

so if we choose a different Hermitian metric on L then $F(h)$ changes by an exact form. This allows us to define the first Chern class of the line bundle L to be

$$c_1(L) = \frac{1}{2\pi}[F(h)] \in H^2(M, \mathbf{R}).$$

The $\partial\bar{\partial}$-lemma and (14) imply that every real (1,1)-form in $c_1(L)$ is the curvature of some Hermitian metric on L.

Remark 1.25. The normalizing factor of 2π is chosen because it turns out that this way $c_1(L)$ is an integral cohomology class. We will not need this, but it is an important fact about characteristic classes. See [22] p. 139.

For us the most important property that a line bundle can have is the positivity of its curvature.

Definition 1.26. Let us call a real (1,1)-form positive, if the symmetric bilinear form $(X,Y) \mapsto \alpha(X,JY)$ defined for real tangent vectors X,Y is positive definite. For instance the Kähler form of a Kähler metric is positive.

A cohomology class in $H^2(M,\mathbf{R})$ is called positive if it can be represented by a closed positive (1,1)-form. Finally we call a line bundle positive if its first Chern class is positive. Equivalently a line bundle is positive if for a suitable Hermitian metric h the curvature form $F(h)$ is a Kähler form.

Example 1.27. The $\mathcal{O}(-1)$ bundle over $\mathbf{CP}^n$ has a natural Hermitian metric h since it is a subbundle of the trivial bundle $\mathbf{CP}^n \times \mathbf{C}^{n+1}$ on which we can use the standard Hermitian metric of $\mathbf{C}^{n+1}$. On the open set U_0 if we use coordinates $z^i = \dfrac{Z_i}{Z_0}$ for $i = 1,\ldots,n$, then a holomorphic section of $\mathcal{O}(-1)$ over U_0 is given by

$$s : (z^1,\ldots,z^n) \mapsto (1, z^1,\ldots,z^n) \in \mathbf{C}^{n+1},$$

since $(z^1,\ldots,z^n) \in U_0$ corresponds to the point $[1 : z^1 : \ldots : z^n]$ in homogeneous coordinates. By Lemma 1.24 the curvature form of h is then

$$F(h) = -\sqrt{-1}\partial\overline{\partial}\log h(s) = -\sqrt{-1}\partial\overline{\partial}\log(1 + |z^1|^2 + \ldots + |z^n|^2),$$

so $F(h) = -\omega_{FS}$ in terms of the Fubini-Study metric of Example 1.11.

The metric h induces a metric on the dual bundle $\mathcal{O}(1)$, whose curvature form will then be ω_{FS}. Since this is a Kähler form, $\mathcal{O}(1)$ is a positive line bundle.

In this course, just as we will restrict our attention to compact complex manifolds which are submanifolds of projective space, we will also restrict attention to Kähler metrics whose Kähler class is the first Chern class of a line bundle. In the next section we will see that if a compact complex manifold admits such a Kähler metric, then it is automatically a projective manifold.

1.7 Line bundles and projective embeddings

Suppose that $L \to M$ is a holomorphic line bundle over a complex manifold M. If $s_0,\ldots,s_k$ are sections of L, then over the set $U \subset M$ where at least one s_i is non-zero, we obtain a holomorphic map

$$\begin{aligned} U &\to \mathbf{CP}^k \\ p &\mapsto [s_0(p) : \ldots : s_k(p)]. \end{aligned} \tag{15}$$

Definition 1.28. A line bundle L over M is very ample, if for suitable sections $s_0, \dots, s_k$ of L the map (15) defines an embedding of M into $\mathbf{CP}^k$. A line bundle L is ample if for a suitable integer $r > 0$ the tensor power L^r is very ample.

Example 1.29. The bundle $\mathcal{O}(1)$ over $\mathbf{CP}^n$ is very ample, and the sections $Z_0, \dots, Z_n$ from Example 1.22 define the identity map from $\mathbf{CP}^n$ to itself. More generally for any projective manifold $V \subset \mathbf{CP}^n$, the restriction of $\mathcal{O}(1)$ to V is a very ample line bundle. Conversely if L is a very ample line bundle over V, then L is isomorphic to the restriction of the $\mathcal{O}(1)$ bundle under a projective embedding furnished by sections of L.

The following is a fundamental result relating the curvature of a line bundle to ampleness.

Theorem 1.30 (Kodaira embedding theorem)**.** *Let L be a line bundle over a compact complex manifold M. Then L is ample if and only if the first Chern class $c_1(L)$ is positive.*

The difficult implication is that a line bundle with positive first Chern class is ample. The proof requires showing that a sufficiently high power of the line bundle admits enough holomorphic sections to give rise to an embedding of the manifold, but it is already non-trivial to show that there is at least one non-zero holomorphic section. One way to proceed is through Kodaira's vanishing theorem for cohomology (see [22] p. 189). Another approach is through studying the Bergman kernel for large powers of the line bundle L, which we will discuss in Section 6.

Example 1.31. Let L be the trivial line bundle over $\mathbf{C}^n$, so holomorphic sections of L are simply holomorphic functions on $\mathbf{C}^n$. Write $\mathbf{1}$ for the section given by the constant function 1. For $k > 0$ let us define the Hermitian metric h so that $h(\mathbf{1}) = e^{-k|z|^2}$. Then by Lemma 1.24,

$$F(h) = k\sqrt{-1}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i.$$

When k is very large, then on the one hand the section $\mathbf{1}$ decays very rapidly as we move away from the origin, and on the other hand the curvature of the line bundle is very large. The idea of Tian's argument [49] which we will explain in Section 6.2 is that if the curvature of a line bundle L at a point p is very large, then using a suitable cutoff function, we can glue the rapidly decaying holomorphic section $\mathbf{1}$ into a neighborhood around p.

Because of the cutoff function this will no longer be holomorphic, but the error is sufficiently small so that it can be corrected to obtain a holomorphic section of L, which is "peaked" at p. If the curvature of the line bundle is large everywhere, then this construction will give rise to enough holomorphic sections to embed the manifold into projective space.

A much simpler result is that if the line bundle L over a compact Kähler manifold is negative, i.e. $-c_1(L)$ is a positive class, then there are no non-zero holomorphic sections at all. To see this, choose a Hermitian metric on L whose curvature form $F_{k\bar{l}}$ is negative definite. From the definition of the curvature, the Chern connection satisfies

$$\nabla_k \nabla_{\bar{l}} = \nabla_{\bar{l}} \nabla_k + F_{k\bar{l}}.$$

If s is a global holomorphic section of L over M, then we have

$$0 = \langle g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} s, s \rangle_h = \langle g^{k\bar{l}} \nabla_{\bar{l}} \nabla_k s, s \rangle_h + g^{k\bar{l}} F_{k\bar{l}} |s|_h^2 \leqslant \langle g^{k\bar{l}} \nabla_{\bar{l}} \nabla_k s, s \rangle_h - c|s|_h^2$$

for some constant $c > 0$, since $F_{k\bar{l}}$ is negative definite. Integrating this over M and integrating by parts we get

$$0 \leqslant -\int_M |\nabla s|^2_{g\otimes h}\, dV - c \int_M |s|_h^2\, dV. \tag{16}$$

Here dV is the volume form of the metric g, and we are writing $g \otimes h$ for the natural Hermitian metric on $T^{1,0} \otimes L$, which in coordinates can be written as

$$|\nabla s|^2_{g\otimes h} = g^{k\bar{l}} h \nabla_k s \overline{\nabla_l s}.$$

From the inequality (16) it is clear that we must have $s = 0$.

It is perhaps instructive to work out the integration by parts carefully to familiarize oneself with the notation. Note first of all that using the Levi-Civita connection together with the Chern connection of L, we obtain natural connections on any vector bundle related to $T^{1,0}M$ and L, and their tensor products, direct sums, etc. Now let us define the vector field $v^{\bar{l}}$ by

$$v^{\bar{l}} = g^{k\bar{l}} h (\nabla_k s) \bar{s}.$$

Note that $g^{j\bar{k}}$ is a section of $T^{1,0}M \otimes T^{0,1}M$, h is a section of $L^* \otimes \overline{L^*}$, $\nabla_k s$ is a section of $\Omega^{1,0}M \otimes L$ and $\bar{s}$ is a section of $\overline{L}$. The section $v^{\bar{l}}$ of $T^{0,1}M$ is obtained by taking the tensor product of these 4 sections, and performing various contractions between pairwise dual spaces. The function $\nabla_{\bar{l}} v^{\bar{l}}$ is the divergence of a vector field, so it has integral zero (it is the exterior derivative

$d(\iota_v dV)$ of the contraction of the volume form with v, so we can use Stokes's theorem). Using the product rule, we have

$$\nabla_{\bar{l}} v^{\bar{l}} = (\nabla_{\bar{l}} g^{k\bar{l}}) h(\nabla_k s)\overline{s} + g^{k\bar{l}}(\nabla_{\bar{l}} h)(\nabla_k s)\overline{s} + g^{k\bar{l}} h(\nabla_{\bar{l}} \nabla_k s)\overline{s} + g^{k\bar{l}} h(\nabla_k s)(\nabla_{\bar{l}} \overline{s}),$$

where each time the covariant derivative of the appropriate bundle is used. By the defining properties of the Levi-Civita and Chern connections, we have $\nabla g = 0$ and $\nabla h = 0$, so

$$\nabla_{\bar{l}} v^{\bar{l}} = g^{k\bar{l}} h(\nabla_{\bar{l}} \nabla_k s)\overline{s} + g^{k\bar{l}} h(\nabla_k s)(\nabla_{\bar{l}} \overline{s}).$$

Integrating this equation gives the integration by parts formula.

1.8 Exercises

Exercise 1.1. In the proof of the $\sqrt{-1}\partial\overline{\partial}$-lemma we used the fact that on a compact Kähler manifold if a $(1,0)$-form α satisfies $\partial\alpha = \partial^*\alpha = 0$, then also $\overline{\partial}\alpha = 0$. Verify this statement by showing that under these assumptions $g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} \alpha_i = 0$ and then integrating by parts. The generalization of this statement is that on a Kähler manifold the ∂ and $\overline{\partial}$-Laplacians coincide (see [22] p. 115).

Exercise 1.2. Show that if (E, h_E) and (F, h_F) are Hermitian holomorphic bundles whose curvature forms are R_E and R_F respectively, then the curvature of the tensor product $(E \otimes F, h_E \otimes h_F)$ is the sum

$$R_E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes R_F,$$

where Id_E and Id_F are the identity endomorphisms of E, F.

Exercise 1.3. Verify the following commutation relations for a (0,1)-vector field $v^{\bar{p}}$ and (0,1)-form $\alpha_{\bar{p}}$:

$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k) v^{\bar{p}} = -R^{\bar{p}}_{\ \bar{q}k\bar{l}} v^{\bar{q}}$$
$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k) \alpha_{\bar{p}} = R^{\bar{q}}_{\ \bar{p}k\bar{l}} \alpha_{\bar{q}}.$$

To understand the signs, note that a Hermitian metric identifies the dual E^* with the conjugate $\overline{E}$. In particular $T^{1,0} \cong \Omega^{0,1}$ and dual to these spaces are $T^{0,1} \cong \Omega^{1,0}$. In addition from Exercise 1.2 the curvature of E^* is the negative of the curvature of E.

Exercise 1.4. Verify the following identities for the curvature of a Kähler metric:

$$R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}}$$
$$\nabla_p R_{i\bar{j}k\bar{l}} = \nabla_i R_{p\bar{j}k\bar{l}}.$$

If they are familiar, compare these to the identities satisfied by the curvature tensor of a Riemannian metric, in particular the first and second Bianchi identities.

Exercise 1.5. Show that a Hermitian metric is Kähler if and only if the Levi-Civita and Chern connections coincide on $T^{1,0}M$.

Exercise 1.6. A holomorphic vector field is a section v^i of $T^{1,0}M$ such that $\nabla_{\bar{k}} v^i = 0$. Show that on a compact Kähler manifold with negative definite Ricci form there are no non-zero holomorphic vector fields.

Exercise 1.7. Let L be a holomorphic line bundle on a compact Kähler manifold with $c_1(L) = 0$. Show that if L is not the trivial line bundle, then it has no non-zero global holomorphic sections.

Exercise 1.8. Show that for the Fubini-Study metric ω_{FS} in Example 1.11, the Ricci form satisfies $\mathrm{Ric}(\omega_{FS}) = (n+1)\omega_{FS}$, i.e. ω_{FS} is a Kähler-Einstein metric. Deduce from this that the canonical line bundle of $\mathbf{CP}^n$ is $K_{\mathbf{CP}^n} = \mathcal{O}(-n-1)$, assuming that every line bundle on $\mathbf{CP}^n$ is of the form $\mathcal{O}(l)$.

2 Analytic preliminaries

In this section we collect some fundamental results about elliptic operators on manifolds, which we will need later on. The most important results for us will be the Schauder estimates Theorem 2.9, and the solution of linear elliptic equations on compact manifolds, Theorem 2.12. The basic reference for elliptic equations of second order is Gilbarg-Trudinger [21]. For analysis on manifolds Aubin [6] gives an overview and Donaldson [13] is also a good resource.

2.1 Harmonic functions on $\mathbf{R}^n$

Let $U \subset \mathbf{R}^n$ be an open set. A function $f : U \to \mathbf{R}$ is called harmonic, if

$$\Delta f := \frac{\partial^2 f}{\partial x^1 \partial x^1} + \ldots + \frac{\partial^2 f}{\partial x^n \partial x^n} = 0 \text{ on } U.$$

For any $x \in \mathbf{R}^n$ let us write $B_r(x)$ for the open r-ball around x. For short we will write $B_r = B_r(0)$. The most basic property of harmonic functions is the following.

Theorem 2.1 (Mean value theorem)**.** *If $f : U \to \mathbf{R}$ is harmonic, $x \in U$ and the r ball $B_r(x) \subset U$, then*

$$f(x) = \frac{1}{\mathrm{Vol}(\partial B_r)} \int_{\partial B_r(x)} f(y)\, dy.$$

Proof. For $\rho \leqslant r$ let us define

$$F(\rho) = \int_{\partial B_1} f(x + \rho y)\, dy.$$

Then

$$\begin{aligned} F'(\rho) &= \int_{\partial B_1} \nabla f(x + \rho y) \cdot y\, dy \\ &= \int_{B_1} \Delta f(x + \rho y)\, dy = 0, \end{aligned}$$

where we used Green's theorem. This means that F is constant, but also by changing variables

$$F(r) = \frac{\mathrm{Vol}(\partial B_1)}{\mathrm{Vol}(\partial B_r)} \int_{\partial B_r(x)} f(y)\, dy,$$

while $\lim_{\rho \to 0} F(\rho) = \mathrm{Vol}(\partial B_1) f(x)$. $\square$

By averaging the mean value property over spheres of different radii, we obtain the following.

Corollary 2.2. *Let $\eta : \mathbf{R}^n \to \mathbf{R}$ be smooth, radially symmetric, supported in B_1, and $\int \eta = 1$. If $f : B_2 \to \mathbf{R}$ is harmonic, then for all $x \in B_1$ we have*

$$f(x) = \int_{\mathbf{R}^n} f(x-y)\eta(y)\,dy = \int_{\mathbf{R}^n} f(y)\eta(x-y)\,dy.$$

An important consequence is that the L^1-norm of a harmonic function on B_2 controls all the derivatives of the function on the smaller ball B_1.

Corollary 2.3. *There are constants C_k such that if $f : B_2 \to \mathbf{R}$ is harmonic, then*

$$\sup_{B_1} |\nabla^k f| \leqslant C_k \int_{B_2} |f(y)|\,dy.$$

In particular even if f is only assumed to be twice differentiable, it follows that f is smooth on B_1.

Proof. For any $x \in B_1$ we can use the previous corollary and differentiate under the integral sign to get

$$\nabla^k f(x) = \int_{\mathbf{R}^n} f(y)\nabla^k \eta(x-y)\,dy,$$

so

$$|\nabla^k f(x)| \leqslant (\sup_{B_1} |\nabla^k \eta|) \int_{B_2} |f(y)|\,dy.$$

The result follows with $C_k = \sup |\nabla^k \eta|$. ☐

This interior regularity result together with a scaling argument implies the following "rigidity" statement.

Corollary 2.4 (Liouville's theorem)**.** *We say that a function $f : \mathbf{R}^n \to \mathbf{R}$ has sub-linear growth, if*

$$\lim_{R\to\infty} R^{-1} \sup_{B_R} |f| = 0.$$

If f is harmonic on $\mathbf{R}^n$ and has sub-linear growth, then f is constant.

Proof. For $r > 0$ let $f_r(x) = f(rx)$, which is also harmonic. The previous corollary implies that

$$|\nabla f_r(0)| \leqslant C_1 \int_{B_2} |f_r(x)|\, dx \leqslant C' \sup_{B_2} |f_r| = C' \sup_{B_{2r}} |f|,$$

for some constant C'. But $\nabla f_r(0) = r\nabla f(0)$, so we get

$$|\nabla f(0)| \leqslant C' r^{-1} \sup_{B_{2r}} |f|$$

for all $r > 0$. Taking $r \to \infty$, this implies $\nabla f(0) = 0$. By translating f, we get $\nabla f(x) = 0$ for all x, so f is constant. $\square$

2.2 Elliptic differential operators

In Riemannian geometry many of the natural differential equations that arise are elliptic. We will focus on scalar equations of second order. A general linear differential operator of second order is of the form

$$L(f) = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 f}{\partial x^j \partial x^k} + \sum_{l=1}^{n} b_l \frac{\partial f}{\partial x^l} + cf, \tag{17}$$

where $f, a_{jk}, b_l, c : \Omega \to \mathbf{R}$ are all functions on an open set $\Omega \subset \mathbf{R}^n$. This operator is elliptic, if the matrix (a_{jk}) is positive definite. In addition we will always assume that this coefficient matrix is symmetric, i.e. $a_{jk} = a_{kj}$.

From now on we will assume that the a_{jk}, b_l, c are all smooth functions. In addition we will usually work on a compact manifold (in which Ω is a coordinate chart), so we will be able to assume the stronger condition of uniform ellipticity:

$$\lambda|v|^2 \leqslant \sum_{j,k=1}^{n} a_{jk}(x) v^j v^k \leqslant \Lambda |v|^2, \quad \text{for all } x \in \Omega$$

for all vectors v and some constants $\lambda, \Lambda > 0$.

While we assume the coefficients of our operator to be smooth, in constructing solutions to linear equations it is usually easiest to first obtain a weak solution. Weak solutions are defined in terms of the formal adjoint L^* of L, which is the operator

$$L^*(f) = \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x^j \partial x^k}(a_{jk} f) - \sum_{l=1}^{n} \frac{\partial}{\partial x^l}(b_l f) + cf.$$

We say that a function f which is locally integrable on Ω is a weak solution of the equation $L(f) = g$ if

$$\int_\Omega f L^*(\varphi)\, dV = \int_\Omega g\varphi\, dV,$$

for all compactly supported smooth functions φ on Ω, where dV is the usual volume measure on $\mathbf{R}^n$. The adjoint is defined so that if f is a weak solution of $L(f) = g$ and f is actually smooth, then integration by parts shows that $L(f) = g$ in the usual sense. A fundamental property of elliptic operators is that weak solutions are automatically smooth.

Theorem 2.5. *Suppose that f is a weak solution of the equation $L(f) = g$, where L is a linear elliptic operator with smooth coefficients and g is a smooth function. Then f is also smooth.*

There are many more general regularity statements, but for us this simple one will suffice. The proof is somewhat involved, and requires techniques that we will not use in the rest of the course. One approach to the proof is to first use convolutions to construct smoothings f_ε of f, and then derive estimates for the f_ε in various Sobolev spaces, which are independent of ε. The Sobolev embedding theorem will then ensure that f, which is the limit of the f_ε as $\varepsilon \to 0$, is smooth. For details of this approach, see for example Griffiths-Harris [22] p. 380.

2.3 Schauder estimates

In Section 2.2 we saw that solutions of elliptic equations have very strong regularity properties. In this section we will see a more refined version of this idea. We will once again work in a domain $\Omega \subset \mathbf{R}^n$. More precisely we should generally work on a bounded open set with at least C^1 boundary. Not much is lost by assuming that Ω is simply an open ball in $\mathbf{R}^n$. Recall that for $\alpha \in (0, 1)$ the C^α Hölder coefficient of a function f on Ω is defined as

$$|f|_{C^\alpha} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Using this we can define the $C^{k,\alpha}$-norms for $k \in \mathbf{N}$ and $\alpha \in (0, 1)$ as

$$\|f\|_{C^{k,\alpha}} = \sup_{|\mathbf{l}|\leqslant k, x\in\Omega} |\partial^{\mathbf{l}} f(x)| + \sup_{|\mathbf{l}|=k} |\partial^{\mathbf{l}} f|_{C^\alpha},$$

where $\mathbf{l} = (l_1, \ldots, l_n)$ is a multi-index, and

$$\partial^{\mathbf{l}} = \frac{\partial}{\partial x^{l_1}} \cdots \frac{\partial}{\partial x^{l_n}}$$

is the corresponding partial derivative of order $|\mathbf{l}| = l_1 + \ldots + l_n$. The space $C^{k,\alpha}(\Omega)$ is the space of functions on Ω whose $C^{k,\alpha}$ norm is finite. If $k > 0$ then such functions are necessarily k-times continuously differentiable. Moreover $C^{k,\alpha}(\Omega)$ is complete, i.e. any Cauchy sequence with respect to the $C^{k,\alpha}$ norm converges in $C^{k,\alpha}$.

Of crucial importance is the following consequence of the Arzela-Ascoli theorem.

Theorem 2.6. *Suppose that Ω is a bounded set and $u_k : \Omega \to \mathbf{R}$ is a sequence of functions such that $\|u_k\|_{C^{k,\alpha}} < C$ for some constant C. Then a subsequence of the u_k is convergent in $C^{l,\beta}$ for any l, β such that $l+\beta < k+\alpha$.*

Let us suppose again that

$$L(f) = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 f}{\partial x^j \partial x^k} + \sum_{l=1}^{n} b_l \frac{\partial f}{\partial x^l} + cf$$

is a uniformly elliptic second order differential operator with smooth coefficients. In particular we have the inequalities

$$\lambda |v|^2 \leqslant \sum_{j,k=1}^{n} a_{jk}(x) v^j v^k \leqslant \Lambda |v|^2, \quad \text{for all } x \in \Omega, \tag{18}$$

for some $\lambda, \Lambda > 0$.

Theorem 2.7 (Local Schauder estimates)**.** *Let Ω be a bounded domain, let $\Omega' \subset \Omega$ be a smaller domain with the distance $d(\Omega', \partial\Omega) > 0$, and suppose that $\alpha \in (0,1)$ and $k \in \mathbf{N}$. There is a constant C such that if $L(f) = g$, then we have*

$$\|f\|_{C^{k+2,\alpha}(\Omega')} \leqslant C(\|g\|_{C^{k,\alpha}(\Omega)} + \|f\|_{C^0(\Omega)}). \tag{19}$$

Moreover C only depends on k, α, the domains Ω, Ω', the $C^{k,\alpha}$-norms of the coefficients of L, and the constants of ellipticity λ, Λ in (18).

Sketch of proof. There are several approaches to the proof, usually reducing the problem to the case when the coefficients of L are constant (see Gilbarg-Trudinger [21]). One approach is to argue by contradiction, using Liouville's theorem for harmonic functions (see Simon [43]). This type of "blow-up" argument is very common in geometric analysis.

We will only treat the case $k = 0$, since the general case can be reduced to this by differentiating the equation k times. Moreover we will only treat operators L with $b_l, c = 0$ since again the general case can be reduced to this one (see Exercise 2.3).

First we show a weaker estimate, namely that under the assumptions of the theorem we have a constant C such that

$$\|f\|_{C^{2,\alpha}(\Omega')} \leqslant C(\|g\|_{C^\alpha(\Omega)} + \|f\|_{C^2(\Omega)}). \tag{20}$$

More precisely for any $x \in \Omega$ we let $d_x = \min\{1, d(x, \partial\Omega)\}$. We will show that for some constant C we have

$$\min\{d_x, d_y\}^\alpha \frac{|\partial^{\mathbf{l}} f(x) - \partial^{\mathbf{l}} f(y)|}{|x-y|^\alpha} \leqslant C(\|g\|_{C^\alpha(\Omega)} + \|f\|_{C^2(\Omega)}), \tag{21}$$

for all $x, y \in \Omega$ and $\partial^{\mathbf{l}}$ any second order partial derivative.

To argue by contradiction let us fix constants K, λ, Λ and suppose that for arbitrary C there exist functions a_{jk}, f, g on Ω satisfying the equation

$$\sum_{j,k} a_{jk} \frac{\partial^2 f}{\partial x^j \partial x^k} = g,$$

such that in addition the a_{jk} satisfy $\|a_{jk}\|_{C^\alpha} \leqslant K$, the uniform ellipticity condition (18) holds, and $\|g\|_{C^\alpha}, \|f\|_{C^2} \leqslant 1$. Moreover there are points $p, q \in \Omega$ and a second order partial derivative $\partial^{\mathbf{l}}$ such that

$$\min\{d_p, d_q\}^\alpha \frac{|\partial^{\mathbf{l}} f(p) - \partial^{\mathbf{l}} f(q)|}{|p-q|^\alpha} = C, \tag{22}$$

and at the same time C is the largest possible value for this expression for other choices of the points and $\mathbf{l}$. For these points let us write

$$\frac{|\partial^{\mathbf{l}} f(p) - \partial^{\mathbf{l}} f(q)|}{|p-q|^\alpha} = M \geqslant C,$$

and $r = |p-q|$. We define the rescaled function $\tilde{f}(x) = M^{-1} r^{-2-\alpha} f(p+rx)$, and let

$$F(x) = \tilde{f}(x) - \tilde{f}(0) - \sum_k x^k \partial_k \tilde{f}(0) - \frac{1}{2} \sum_{j,k} x^j x^k \partial_j \partial_k \tilde{f}(0).$$

Then this function F satisfies the following properties:

(i) F is defined on (at least) a ball of radius d_p/r around the origin.

(ii) $F(0) = \partial F(0) = \partial^2 F(0) = 0$, where ∂^2 means any second order derivative.

(iii) On the ball of radius $d_p/(2r)$ around the origin, we have $|\partial^2 F|_{C^\alpha} \leqslant 2^\alpha$.

(iv) For $y = r^{-1}(q-p)$ we have $|y| = 1$ and

$$|\partial^{\mathbf{l}} F(0) - \partial^{\mathbf{l}} F(y)| = 1.$$

(v) F satisfies the equation

$$\sum_{j,k} a_{jk}(p+rx)\frac{\partial^2 F(x)}{\partial x^j \partial x^k} = M^{-1}r^{-\alpha}(g(p+rx) - g(p))$$

$$+M^{-1}r^{-\alpha}\sum_{j,k}(a_{jk}(p) - a_{jk}(p+rx))\frac{\partial^2 f(p)}{\partial x^j \partial x^k}.$$

Now suppose that we can perform this construction for larger and larger C, obtaining a sequence of functions $F^{(i)}$ as above, together with $a_{jk}^{(i)}$, $g^{(i)}$, unit vectors $y^{(i)}$ and second order partials $\partial^{\mathbf{l}_i}$. After choosing a subsequence we can assume that the $y^{(i)}$ converge to a vector y and the second order partial derivatives are all the same $\partial^{\mathbf{l}}$. Since we have assumed that $\|f\|_{C^2} \leqslant 1$, from (22) we see that $rd_p^{-1} \to 0$ as $C \to \infty$, so the $F^{(i)}$ are defined on larger and larger balls. From properties (ii) and (iii) the $F^{(i)}$ satisfy uniform $C^{2,\alpha}$ bounds on fixed balls, so on each fixed ball we can extract a convergent subsequence in C^2. By a diagonal argument we obtain a function $G : \mathbf{R}^n \to \mathbf{R}$ which on each fixed ball is a C^2-limit of a subsequence of the $F^{(i)}$, and in particular G satisfies the conditions (i)-(iv). The "stretched" functions $x \mapsto a_{jk}^{(i)}(p+rx)$ satisfy uniform C^α bounds, so by choosing a further subsequence we can assume that they converge to functions A_{jk}, which because of the stretching are actually constant. The uniform C^α bounds on the $g^{(i)}$ and $a_{jk}^{(i)}$, the assumption that $\|f\|_{C^2} \leqslant 1$, together with property (v) imply that

$$\sum_{j,k} A_{jk}\frac{\partial^2 G}{\partial x^j \partial x^k} = 0.$$

After a linear change of coordinates by a matrix T, we obtain a harmonic function $H(x) = G(Tx)$ defined on all of $\mathbf{R}^n$. By Corollary 2.3 or Theorem 2.5 the function H is smooth, and so its second derivative $\partial^{\mathbf{l}} H$ are also harmonic. From properties (ii) and (iii) we have $|\partial^{\mathbf{l}} G(x)| \leqslant 2^{\alpha}|x|^{\alpha}$, so each second derivative of H is a harmonic function with sub-linear growth, which is therefore constant by Corollary 2.4. This implies that $\partial^{\mathbf{l}} G$ is identically zero, contradicting property (iv). This proves the estimate (20).

From (21) one can deduce the estimate we need (replacing the C^2-norm of f by the C^0-norm), by using a more refined interpolation inequality than the one in Exercise 2.2 (see [21]). Alternatively we can proceed with another argument by contradiction as follows. Still under the same assumptions as in the statement of the theorem, we will now show that there is a constant C such that

$$d_x^2 |\partial^{\mathbf{l}} f(x)| \leqslant C(\|g\|_{C^{\alpha}(\Omega)} + \|f\|_{C^0(\Omega)}),$$

for all $x \in \Omega$ and second order derivative $\partial^{\mathbf{l}}$. We use a very similar argument to before. Suppose that $\|g\|_{C^{\alpha}}, \|f\|_{C^0} \leqslant 1$. Choose $p \in \Omega$, and a multi-index $\mathbf{l}$ such that

$$d_p^2 |\partial^{\mathbf{l}} f(p)| = C,$$

and C is the largest possible value of this expression. Let

$$|\partial^{\mathbf{l}} f(p)| = M \geqslant C.$$

Define the rescaled function $F(x) = M^{-1} d_p^{-2} f(p + d_p x)$. Then F satisfies

(i) F is defined at least on a ball of radius 1 around the origin.

(ii) On the ball of radius $1/2$ around the origin we have $\|F\|_{C^2} \leqslant K$ for some fixed constant K.

(iii) $|\partial^{\mathbf{l}} F(0)| = 1$.

(iv) F satisfies the equation

$$\sum_{j,k} a_{jk}(p + d_p x) \frac{\partial^2 F(x)}{\partial x^j \partial x^k} = M^{-1} g(p + d_p x).$$

(v) $|F| \leqslant M^{-1} d_p^{-2} \|f\|_{C^0}$.

If we have a family of such functions $F^{(i)}$ with larger and larger C, then since $d_p \leqslant 1$, and $d_p^2 M = C$, the coefficients and right hand sides of the equation in property (iv) will satisfy uniform C^{α} bounds. It follows from our previous estimate (20) and property (ii) that the functions $F^{(i)}$ satisfy uniform $C^{2,\alpha}$ bounds on the ball of radius $1/4$ around the origin. A subsequence will then converge in C^2 to a limiting function G on $B_{1/4}$, with $|\partial^{\mathbf{l}} G(0)| = 1$ for some second order partial derivative $\partial^{\mathbf{l}}$ by property (iii). But property (v) and the fact that $d_p^2 M = C \to \infty$ implies that G is identically zero on $B_{1/4}$, which is a contradiction. $\square$

We will sometimes need the following strengthening of this estimate.

Proposition 2.8. *Under the same conditions as the previous theorem, we actually have a constant C such that*

$$\|f\|_{C^{k+2,\alpha}(\Omega')} \leqslant C(\|L(f)\|_{C^{k,\alpha}(\Omega)} + \|f\|_{L^1(\Omega)}).$$

To prove this, one just needs to show that under the conditions of Theorem 2.7, the C^0-norm of f is controlled by the L^1-norm of f together with the C^α-norm of Lf. This can be done by using a blow-up argument similar to what we have used above (see Exercise 2.4), although the more standard way is to use similar estimates in Sobolev spaces, together with the Sobolev embedding theorem. Note that in the special case when $Lf = 0$, this estimate generalizes the basic interior estimate Corollary 2.3 for harmonic functions.

An important point which does not follow from our arguments is that we do not need to know a priori that $f \in C^{k+2,\alpha}$. In other words if we just know that $f \in C^2$, so that the equation $L(f) = g$ makes sense, then if the coefficients of L and g are in $C^{k,\alpha}$, it follows that $f \in C^{k+2,\alpha}$ and the inequality (19) holds. For this one needs to work harder, see Gilbarg-Trudinger [21], Chapter 6.

On a smooth manifold M the Hölder spaces can be defined locally in coordinate charts. More precisely we cover M with coordinate charts U_i. Then any tensor T on M can be written in terms of its components on each U_i. The $C^{k,\alpha}$-norm of the tensor T can be defined as the supremum of the $C^{k,\alpha}$-norms of the components of T over each coordinate chart.

This works well if there are finitely many charts, which we can achieve if M is compact for example. It is more natural, however, to work on Riemannian manifolds, and define the Hölder norms relative to the metric. If (M, g) is a Riemannian manifold, then we can use parallel translation along geodesics with respect to the Levi-Civita connection to compare tensors at different points. For a tensor T we can define

$$|T|_{C^\alpha} = \sup_{x,y} \frac{|T(x) - T(y)|}{d(x,y)^\alpha},$$

where the supremum is taken over those pairs of points x, y which are connected by a unique minimal geodesic. The difference $T(x) - T(y)$ is computed by parallel transporting $T(y)$ to x along this minimal geodesic. We then define

$$\|f\|_{C^{k,\alpha}} = \sup_M (|f| + |\nabla f| + \ldots + |\nabla^k f|) + |\nabla^k f|_{C^\alpha}.$$

These Hölder norms are uniformly equivalent to the norms defined using charts, as long as we only have finitely many charts.

A linear second order elliptic operator on a smooth manifold is an operator which in each local chart can be written as (17), where (a_{jk}) is symmetric and positive definite. The local Schauder estimates of Theorem 2.7 can easily be used to deduce global estimates on a compact manifold. In fact if we cover the manifold by coordinate charts U_i, then we will get estimates on slightly smaller open sets U_i', but we can assume that these still cover the manifold. We therefore obtain the following.

Theorem 2.9 (Schauder estimates)**.** *Let (M, g) be a compact Riemannian manifold, and L a second order elliptic operator on M. For any k and $\alpha \in (0, 1)$ there is a constant C, such that*

$$\|f\|_{C^{k+2,\alpha}(M)} \leqslant C(\|L(f)\|_{C^{k,\alpha}(M)} + \|f\|_{L^1(M)}),$$

where C depends on (M, g), k, α, the $C^{k,\alpha}$-norms of the coefficients of L, and the constants of ellipticity λ, Λ in (18). As we mentioned above, it is enough to assume that $f \in C^2$, and it follows that actually $f \in C^{k+2,\alpha}$ whenever $L(f)$ and the coefficients of L are in $C^{k,\alpha}$.

This theorem has the important consequence that the solution spaces of linear elliptic equations on compact manifolds are finite dimensional. In particular see Exercise 2.6.

Corollary 2.10. *Let L be a second order elliptic operator on a compact Riemannian manifold M. Then the kernel of L*

$$\ker L = \{f \in L^2(M) \,|\, f \text{ is a weak solution of } Lf = 0\}$$

is a finite dimensional space of smooth functions.

Proof. We know from Theorem 2.5, applied locally in coordinate charts, that any weak solution of $Lf = 0$ is actually smooth. To prove that $\ker L$ is finite dimensional we will prove that the closed unit ball in $\ker L$ with respect to the L^2 metric is compact. Indeed, let $f_k \in \ker L$ be a sequence of functions such that $\|f_k\|_{L^2(M)} \leqslant 1$. By Hölder's inequality we then have

$\|f_k\|_{L^1(M)} \leqslant C_1$ for some constant C_1. Applying the Schauder estimates we obtain a constant C_2 such that

$$\|f_k\|_{C^{2,\alpha}(M)} \leqslant C_2.$$

It follows that a subsequence of the f_k converge in C^2 to a function f. Since the convergence is in C^2, we have $f \in \ker L$, and also $\|f\|_{L^2(M)} \leqslant 1$. This shows that any sequence in the unit ball of $\ker L$ has a convergent subsequence, so this ball is compact. Thus $\ker L$ must be finite dimensional. □

2.4 The Laplace operator on Kähler manifolds

The Laplace operator is the fundamental second order differential operator on a Riemannian manifold. On Kähler manifolds we will use one half of the usual Riemannian Laplacian, which can be written in terms of local holomorphic coordinates as

$$\Delta f = g^{k\bar{l}} \nabla_k \nabla_{\bar{l}} f = g^{k\bar{l}} \partial_k \partial_{\bar{l}} f.$$

Recall that $\nabla_k(\partial/\partial\bar{z}^l) = 0$, so the expression using partial derivatives holds even if we are not using normal coordinates, in contrast to the Riemannian case. Rewriting the operator in local real coordinates, we find that the Laplace operator is elliptic.

A useful way to think of the Laplacian is as the operator $\Delta = -\overline{\partial}^*\overline{\partial}$, where

$$\overline{\partial}^* : \Omega^{0,1}M \to C^\infty(M)$$

is the formal adjoint of $\overline{\partial}$. If our manifold is compact, then this means that for any $(0,1)$-form α and function f we have

$$\int_M \langle \alpha, \overline{\partial} f \rangle \, dV = \int_M \langle \overline{\partial}^* \alpha, f \rangle \, dV, \tag{23}$$

where $\langle \cdot, \cdot \rangle$ is the natural Hermitian form induced by the metric, and dV is the Riemannian volume form, so $dV = \dfrac{\omega^n}{n!}$ (see Exercise 2.5). So in local coordinates

$$\langle \alpha, \overline{\partial} f \rangle = g^{k\bar{l}} \alpha_{\bar{l}} \overline{\partial_{\bar{k}} f} = g^{k\bar{l}} \alpha_{\bar{l}} \partial_k \overline{f},$$

while $\langle \overline{\partial}^* \alpha, f \rangle$ is just the product $(\overline{\partial}^* \alpha)\overline{f}$. An integration by parts shows that the relation (23) implies that $\overline{\partial}^* \alpha = -g^{k\bar{l}} \nabla_k \alpha_{\bar{l}}$, and so $-\overline{\partial}^*\overline{\partial}$ agrees with our operator Δ. Note that by using covariant derivatives we do not have to worry about differentiating the metric when we integrate by parts, and at the same time remember that $\partial_k \overline{f} = \nabla_k \overline{f}$.

The same idea works for arbitrary (p,q) forms, giving rise to the Hodge Laplacian $\Delta = -\overline{\partial}^* \overline{\partial} - \overline{\partial}\overline{\partial}^*$ (the term $\overline{\partial}\overline{\partial}^*$ is zero on functions). We can also do the same with ∂ instead of $\overline{\partial}$ and on Kähler manifolds both give rise to the same Laplace operator.

The following existence result for the Poisson equation illustrates a typical method for solving linear elliptic equations.

Theorem 2.11. *Suppose that (M, ω) is a compact Kähler manifold, and let $\rho : M \to \mathbf{R}$ be smooth such that*

$$\int_M \rho \, dV = 0. \tag{24}$$

Then there exists a smooth function $f : M \to \mathbf{R}$ such that $\Delta f = \rho$ on M. (The condition (24) is necessary since an integration by parts shows that Δf has zero integral for any f.)

Sketch of proof. One approach to the proof is to solve a variational problem. Namely we look for a function f minimizing the functional

$$E(f) = \int_M \left(\frac{1}{2} |\nabla f|^2 + \rho f \right) dV,$$

subject to the constraint $\int_M f \, dV = 0$. Here $|\nabla f|^2 = g^{k\bar{l}} \nabla_k f \nabla_{\bar{l}} f$. A suitable function space to work on is the space L^2_1 of functions which have one weak derivative in L^2. Alternatively L^2_1 is the completion of the space of smooth functions on M with respect to the norm

$$\|f\|_{L^2_1} = \int_M (|\nabla f|^2 + |f|^2) \, dV.$$

Using the Poincaré inequality one shows that there are constants ε, C such that

$$E(f) \geqslant \varepsilon \|f\|_{L^2_1} - C$$

for all f with zero mean. A minimizing sequence is therefore bounded in L^2_1, so a subsequence will be weakly convergent in L^2_1 to a function F. The lower semicontinuity of the L^2_1-norm implies that F will be a minimizer of E, and the weak convergence shows that $\int_M F \, dV = 0$. Now considering the variation of E at this minimizer F we find that F is a weak solution of $\Delta F = \rho$ (the condition 24 is used here). Finally Theorem 2.5 implies that F is actually smooth. $\square$

With more work, and some tools from functional analysis such as Fredholm operators, one can obtain the following quite general theorem, which describes the mapping properties of linear elliptic operators between Hölder spaces on compact manifolds.

Theorem 2.12. *Let L be an elliptic second order operator with smooth coefficients on a compact Riemannian manifold M. For $k \geqslant 0$ and $\alpha \in (0,1)$ suppose that $\rho \in C^{k,\alpha}(M)$, and $\rho \perp \ker L^*$ with respect to the L^2 product. Then there exists a unique $f \in C^{k+2,\alpha}$ with $f \perp \ker L$ such that $Lf = \rho$. In other words, L is an isomorphism*

$$L : (\ker L)^{\perp} \cap C^{k+2,\alpha} \to (\ker L^*)^{\perp} \cap C^{k,\alpha}.$$

2.5 Exercises

Exercise 2.1. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a harmonic function, such that for some constant C we have $|f(x)| \leqslant C(1+|x|)^k$ for all x. Show that f is a polynomial of degree at most k.

Exercise 2.2. Let $\Omega' \subset \Omega$ be domains as in Theorem 2.7. Show that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that any $f \in C^2(\Omega)$ satisfies

$$\|f\|_{C^1(\Omega')} \leqslant \varepsilon \|f\|_{C^2(\Omega)} + C(\varepsilon)\|f\|_{C^0(\Omega)}.$$

There are more general such interpolation inequalities for Hölder norms, see Gilbarg-Trudinger [21], Section 6.8.

Exercise 2.3. Deduce Theorem 2.7 for general operators L and for general k from the special case when $k = 0$, and $b_l = c = 0$ in the expression (17) for L.

Exercise 2.4. Use a blow-up argument similar to the one we used to prove Theorem 2.7, or an interpolation inequality similar to Exercise 2.2, to prove that under the same conditions as Theorem 2.7, there is a constant C such that

$$\|f\|_{C^0(\Omega')} \leqslant C(\|L(f)\|_{C^\alpha(\Omega)} + \|f\|_{L^1(\Omega)}).$$

Exercise 2.5. Show that on a Kähler manifold (M, ω) of complex dimension n, the Riemannian volume form is given by $\dfrac{\omega^n}{n!}$, where $\omega^n = \omega \wedge \ldots \wedge \omega$.

Exercise 2.6. Given a holomorphic line bundle L over a compact Kähler manifold M, show that the space of global holomorphic sections $H^0(M, L)$ is finite dimensional.

3 Kähler-Einstein metrics

Recall that a Riemannian metric is Einstein, if its Ricci tensor is proportional to the metric. In this section, we are interested in Kähler metrics which are also Einstein. In other words we would like to find Kähler metrics ω which satisfy the equation

$$\mathrm{Ric}(\omega) = \lambda\omega,$$

for some $\lambda \in \mathbf{R}$. By rescaling the metric, we can assume that we are in one of three cases, depending on the sign of λ:

$$\mathrm{Ric}(\omega) = -\omega, \quad \mathrm{Ric}(\omega) = 0, \text{ or } \quad \mathrm{Ric}(\omega) = \omega.$$

As we have seen, the Ricci form of a Kähler metric defines a characteristic class of the manifold, namely

$$c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}(\omega)],$$

which is independent of the Kähler metric ω on M. It follows that in order to find a Kähler-Einstein metric on M, the class $c_1(M)$ must either be a negative, zero, or positive cohomology class. In addition if $c_1(M)$ is positive or negative, then we can only hope to find an Einstein metric in a Kähler class proportional to $c_1(M)$. The main goal of this section is to study the case of a compact Kähler manifold M, with $c_1(M) < 0$. In this case there exists a Kähler-Einstein metric on M, according to the theorem of Aubin and Yau.

We will also briefly discuss the cases when the first Chern class is zero, or positive. When $c_1(M) = 0$, then Yau's theorem says that every Kähler class contains a Kähler-Einstein metric (which is necessarily Ricci flat). The case $c_1(M) > 0$ is still an open problem, although a great deal of progress has been made. The algebro-geometric obstructions in this case will be our subject of study in the remainder of this course.

The basic reference for this section is Yau [55], but there are many places where this proof is explained, for instance Siu [44], Tian [52] or Błocki [7].

3.1 The strategy

Our goal is to prove the following theorem.

Theorem 3.1 (Aubin-Yau)**.** *Let M be a compact Kähler manifold with $c_1(M) < 0$. Then there is a unique Kähler metric $\omega \in -2\pi c_1(M)$ such that $\mathrm{Ric}(\omega) = -\omega$.*

There are lots of manifolds with $c_1(M) < 0$ (see Exercise 3.3), so using this theorem it is possible to construct many examples of Einstein manifolds.

First we rewrite the equation in terms of Kähler potentials. Let ω_0 be any Kähler metric in the class $-2\pi c_1(M)$. By the $\partial\overline{\partial}$-lemma there is a smooth function F on M, such that

$$\mathrm{Ric}(\omega_0) = -\omega_0 + \sqrt{-1}\partial\overline{\partial}F. \tag{25}$$

If $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ is another Kähler metric in the same class, then

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\omega_0) - \sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\omega_0^n},$$

so in order to make sure that $\mathrm{Ric}(\omega) = -\omega$, we need

$$-\sqrt{-1}\partial\overline{\partial}\varphi = \sqrt{-1}\partial\overline{\partial}F - \sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\omega_0^n}.$$

This will certainly be the case if we solve the equation

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F+\varphi}\omega_0^n. \tag{26}$$

At this point we can deal with the uniqueness statement in Theorem 3.1.

Lemma 3.2. *On a compact Kähler manifold M there exists at most one metric $\omega \in -2\pi c_1(M)$ such that $\mathrm{Ric}(\omega) = -\omega$.*

Proof. This is a simple application of the maximum principle. Suppose that $\mathrm{Ric}(\omega_0) = -\omega_0$, so in (25) above we can take $F = 0$. If $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ also satisfies $\mathrm{Ric}(\omega) = -\omega$, then from (26) we get

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{\varphi}\omega_0^n.$$

Suppose that φ achieves its maximum at $p \in M$. In local coordinates at p we have

$$\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) = e^{\varphi}\det(g_{j\bar{k}}),$$

but at p the matrix $\partial_j\partial_{\bar{k}}\varphi$ is negative semi-definite, so

$$\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi)(p) \leqslant \det(g_{j\bar{k}})(p).$$

It follows that $\varphi(p) \leqslant 0$. Since we assumed that φ achieves its maximum at p, we have $\varphi(x) \leqslant 0$ for all x. Looking at the minimum point of φ we similarly find that $\varphi(x) \geqslant 0$ for all x, so we must have $\varphi = 0$. It follows that $\omega = \omega_0$. ∎

We will solve the equation using the continuity method. This involves introducing a family of equations depending on a parameter t, which for $t = 1$ gives the equation we want to solve, but for $t = 0$ simplifies to a simpler equation. We use the family

$$\begin{aligned} (\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n &= e^{tF+\varphi}\omega_0^n, \\ \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi &\text{ is a Kähler form} \end{aligned} \tag{$(*)_t$}$$

for $t \in [0, 1]$. The proof of Theorem 3.1 then comprises of 3 steps:

1. We can solve $(*)_0$. This is clear since $\varphi = 0$ is a solution of $(*)_0$.
2. If $(*)_t$ has a solution for some $t < 1$, then for all sufficiently small $\varepsilon > 0$ we can also solve $(*)_{t+\varepsilon}$. This will be a consequence of the implicit function theorem.
3. If for some $s \in (0, 1]$ we can solve $(*)_t$ for all $t < s$, then we can also solve $(*)_s$. This is the heart of the matter, requiring estimates for the solutions in Hölder spaces, to ensure that we can take a limit along a subsequence as $t \to s$.

Given these 3 statements, it follows that we can solve $(*)_1$, proving Theorem 3.1. We now prove Statement 2.

Lemma 3.3. *Suppose that $(*)_t$ has a smooth solution for some $t < 1$. Then for all sufficiently small $\varepsilon > 0$ we can also find a smooth solution of $(*)_{t+\varepsilon}$.*

Proof. Let us define the operator

$$\begin{aligned} F : C^{3,\alpha}(M) \times [0, 1] &\to C^{1,\alpha}(M) \\ (\psi, t) &\mapsto \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\omega_0^n} - \varphi - tF. \end{aligned}$$

By our assumption we have a smooth function φ_t such that $F(\varphi_t, t) = 0$, and $\omega_t = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_t$ is a Kähler form. We use this Kähler metric ω_t to define the Hölder norms on M. In order to apply the implicit function theorem, we need to compute the derivative of F in the φ direction, at the point (φ_t, t):

$$DF_{(\varphi_t,t)}(\psi, 0) = \frac{n\sqrt{-1}\partial\overline{\partial}\psi \wedge \omega_t^{n-1}}{\omega_t^n} - \psi = \Delta_t\psi - \psi,$$

where Δ_t is the Laplacian with respect to ω_t. Let us write $L(\psi) = \Delta_t\psi - \psi$. This linear operator has trivial kernel: if $\Delta_t\psi - \psi = 0$, then necessarily $\psi = 0$ since

$$\int_M |\psi|^2 \, dV_t = \int_M \psi\Delta_t\psi \, dV_t = -\int_M |\nabla\psi|_t^2 \, dV_t \leqslant 0,$$

where we have put the t subscripts to indicate that everything is computed with respect to ω_t. The operator L is also self-adjoint, so L^* has trivial kernel. It follows from Theorem 2.12 that L is an isomorphism

$$L : C^{3,\alpha}(M) \to C^{1,\alpha}(M).$$

The implicit function theorem then implies that for s sufficiently close to t there exist functions $\varphi_s \in C^{3,\alpha}(M)$ such that $F(\varphi_s, s) = 0$. For s sufficiently close to t this φ_s will be close enough to φ_t in $C^{3,\alpha}$ to ensure that $\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_s$ is a positive form.

What remains for us to show, is that φ_s is actually smooth. We know that

$$\log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_s)^n}{\omega_0^n} - \varphi_s - sF = 0.$$

In local coordinates, if ω_0 has components $g_{j\bar{k}}$, then we can write the equation as

$$\log\det\left(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi_s\right) - \log\det(g_{j\bar{k}}) - \varphi_s - sF = 0.$$

Since we already have $\varphi_s \in C^{3,\alpha}$, we can differentiate the equation, with respect to z^l, say. We get

$$(g_s)^{j\bar{k}}\left(\partial_l g_{j\bar{k}} + \partial_l\partial_j\partial_{\bar{k}}\varphi_s\right) - \partial_l \log\det(g_{j\bar{k}}) - \partial_l\varphi_s - s\partial_l F = 0,$$

where $(g_s)^{j\bar{k}}$ is the inverse of the metric $(g_s)_{j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi_s$, and we are using summation convention. Rewriting this equation,

$$(g_s)^{j\bar{k}}\partial_j\partial_{\bar{k}}(\partial_l\varphi_s) - \partial_l\varphi_s = s\partial_l F + \partial_l\log\det(g_{j\bar{k}}) - (g_s)^{j\bar{k}}\partial_l g_{j\bar{k}}.$$

We think of this as a linear elliptic equation $E(\partial_l\varphi_s) = h$ for the function $\partial_l\varphi_s$, where

$$h = s\partial_l F + \partial_l \log\det(g_{j\bar{k}}) - (g_s)^{j\bar{k}}\partial_l g_{j\bar{k}}.$$

Since $\varphi_s \in C^{3,\alpha}$, the coefficients of the operator E are in $C^{1,\alpha}$, and $h \in C^{1,\alpha}$. It follows that $\partial_l\varphi_s \in C^{3,\alpha}$. Similarly $\partial_{\bar{l}}\varphi_s \in C^{3,\alpha}$ so it follows that $\varphi_s \in C^{4,\alpha}$. Repeating the same argument, we get that $\varphi_s \in C^{5,\alpha}$, and inductively we find that φ_s is actually smooth. This technique of linearizing the equation and obtaining better and better regularity is called bootstrapping.

An alternative approach would be to use the implicit function theorem in $C^{k,\alpha}$ for larger and larger k, and the uniqueness of the solution will imply that the φ_s we obtain is actually smooth. □

The main difficulty is in proving the 3rd statement. For this we need the following proposition.

Proposition 3.4. *There exists a constant $C > 0$ depending only on M, ω_0 and F, such that if φ_t satisfies $(*)_t$ for some $t \in [0,1]$, then*

$$(g_{j\bar{k}} + \partial_j \partial_{\bar{k}} \varphi_t) > C^{-1}(g_{j\bar{k}}),$$

where $g_{j\bar{k}}$ are the components of ω_0 in local coordinates, and the inequality for matrices means that the difference is positive definite. In addition

$$\|\varphi_t\|_{C^{3,\alpha}(M)} \leqslant C,$$

where the Hölder norm is measured with respect to the metric ω_0.

We will prove this in the next two sections. For now we will show how it implies the 3rd statement in the strategy.

Lemma 3.5. *Assume Proposition 3.4. Suppose that $s \in (0,1]$ and that we can solve $(*)_t$ for all $t < s$. Then we can also solve $(*)_s$.*

Proof. Take a sequence of numbers $t_i < s$ such that $\lim t_i = s$. This gives rise to a sequence of functions φ_i which satisfy

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_i)^n = e^{t_i F + \varphi_i}\omega_0^n. \tag{27}$$

Proposition 3.4 implies that the φ_i are uniformly bounded in $C^{3,\alpha}$, so by Theorem 2.6, after choosing a subsequence we can assume that the φ_i converge to a function φ in $C^{3,\alpha'}$ for some $\alpha' < \alpha$. This convergence is strong enough that we can take a limit of the equations (27), so we obtain

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{sF+\varphi}\omega_0^n.$$

In addition Proposition 3.4 implies that the metrics $\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_i$ are all bounded below by a fixed positive definite metric, so the limit $\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$ is also positive definite.

Now the same argument as in the proof of Lemma 3.3 can be used to prove that φ is actually smooth. Alternatively Proposition 3.4 could be strengthened to give uniform bounds on the $C^{k,\alpha}$-norms of φ_t for all k, and then repeating the previous argument (combined with uniqueness) we would obtain a smooth solution. □

3.2 The C^0 and C^2 estimates

What remains is to prove Proposition 3.4. To simplify notation, we will write the equation as

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F+\varphi}\omega^n, \tag{28}$$

and we write $g_{j\bar{k}}$ for the components of the metric ω in local coordinates. We will later apply the results with tF replacing F.

Lemma 3.6. *If φ satisfies the equation (28), then $\sup_M |\varphi| \leqslant \sup_M |F|$.*

Proof. This is essentially the same argument as the uniquenes statement, Lemma 3.2. Suppose that φ achieves its maximum at $p \in M$. Then in local coordinates, the matrix $\partial_j\partial_{\bar{k}}\varphi$ is negative semi-definite at p, so

$$\det(g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi)(p) \leqslant \det(g_{j\bar{k}})(p).$$

Using the equation (28) we get $F(p) + \varphi(p) \leqslant 0$, so $\varphi(p) \leqslant -F(p)$. Since φ is maximal at p, this means that

$$\sup_M \varphi \leqslant -F(p) \leqslant \sup_M |F|.$$

Similarly looking at the minimum point of φ shows that $\sup_M |\varphi| \leqslant \sup_M |F|$. □

Next we would like to find an estimate for the second derivatives of φ. In fact we obtain something weaker, namely an estimate for $\Delta\varphi$, which will imply bounds for the mixed partial derivatives $\partial_j\partial_{\bar{k}}\varphi$. It will be useful to write

$$g'_{j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi,$$

so then

$$g^{j\bar{k}}g'_{j\bar{k}} = n + \Delta\varphi.$$

One more useful notation is to write $\mathrm{tr}_g g' = g^{j\bar{k}}g'_{j\bar{k}}$, and $\mathrm{tr}_{g'} g = g'^{j\bar{k}}g_{j\bar{k}}$. We will also write Δ' for the Laplacian with respect to the metric g'. The key calculation is the following.

Lemma 3.7. *There exists a constant B depending on M and g such that*

$$\Delta' \log \mathrm{tr}_g g' \geqslant -B\mathrm{tr}_{g'} g - \frac{g^{j\bar{k}}R'_{j\bar{k}}}{\mathrm{tr}_g g'},$$

where $R'_{j\bar{k}}$ is the Ricci curvature of g'.

Proof. We will compute in normal coordinates for the metric g around a point $p \in M$. In addition we can assume that g' is diagonal at p, since any Hermitian matrix can be diagonalized by a unitary transformation. In particular at the point p we have

$$\mathrm{tr}_g g' = \sum_i g'_{i\bar{i}}, \quad \mathrm{tr}_{g'} g = \sum_j g'^{j\bar{j}} = \sum_j \frac{1}{g'_{j\bar{j}}}.$$

We can then compute that at p

$$\begin{aligned}
\Delta' \mathrm{tr}_g g' &= g'^{p\bar{q}} \partial_p \partial_{\bar{q}} (g^{j\bar{k}} g'_{j\bar{k}}) \\
&= g'^{p\bar{q}} (\partial_p \partial_{\bar{q}} g^{j\bar{k}}) g'_{j\bar{k}} + g'^{p\bar{q}} g^{j\bar{k}} \partial_p \partial_{\bar{q}} g'_{j\bar{k}} \\
&= g'^{p\bar{q}} (\partial_p \partial_{\bar{q}} g^{j\bar{k}}) g'_{j\bar{k}} - g'^{p\bar{q}} g^{j\bar{k}} R'_{j\bar{k}p\bar{q}} + g'^{p\bar{q}} g^{j\bar{k}} g'^{a\bar{b}} (\partial_j g'_{p\bar{b}})(\partial_{\bar{k}} g'_{a\bar{q}}).
\end{aligned}$$

Using that g' is diagonal, we have

$$\begin{aligned}
g'^{p\bar{q}} (\partial_p \partial_{\bar{q}} g^{j\bar{k}}) g'_{j\bar{k}} &= \sum_{p,j} g'^{p\bar{p}} g'_{j\bar{j}} \partial_p \partial_{\bar{p}} g^{j\bar{j}} \\
&\geqslant -B \sum_{p,j} g'^{p\bar{p}} g'_{j\bar{j}} \\
&= -B (\mathrm{tr}_{g'} g)(\mathrm{tr}_g g'),
\end{aligned}$$

where B is the largest of the numbers $-\partial_p \partial_{\bar{p}} g^{j\bar{j}}$ (more geometrically $-B$ is a lower bound for the bisectional curvature of g). We also have $g'^{p\bar{q}} R'_{j\bar{k}p\bar{q}} = R'_{j\bar{k}}$, so

$$\Delta' \mathrm{tr}_g g' \geqslant -B(\mathrm{tr}_{g'} g)(\mathrm{tr}_g g') - g^{j\bar{k}} R'_{j\bar{k}} + \sum_{p,j,a} g'^{p\bar{p}} g'^{a\bar{a}} |\partial_j g'_{p\bar{a}}|^2. \tag{29}$$

Incorporating the logarithm, we have

$$\begin{aligned}
\Delta' \log \mathrm{tr}_g g' &= \frac{\Delta' \mathrm{tr}_g g'}{\mathrm{tr}_g g'} - \frac{g'^{p\bar{q}} (\partial_p \mathrm{tr}_g g')(\partial_{\bar{q}} \mathrm{tr}_g g')}{(\mathrm{tr}_g g')^2} \\
&\geqslant -B \mathrm{tr}_{g'} g - \frac{g^{j\bar{k}} R'_{j\bar{k}}}{\mathrm{tr}_g g'} + \frac{1}{\mathrm{tr}_g g'} \sum_{p,j,a} g'^{p\bar{p}} g'^{a\bar{a}} |\partial_j g'_{p\bar{a}}|^2 \qquad (30) \\
&\quad - \frac{1}{(\mathrm{tr}_g g')^2} \sum_{p,a,b} g'^{p\bar{p}} (\partial_p g'_{a\bar{a}})(\partial_{\bar{p}} g'_{b\bar{b}})
\end{aligned}$$

Now using the Cauchy-Schwarz inequality twice we have

$$\begin{aligned}\sum_{p,a,b} g'^{p\bar{p}}(\partial_p g'_{a\bar{a}})(\partial_{\bar{p}} g'_{b\bar{b}}) &= \sum_{a,b}\sum_p \sqrt{g'^{p\bar{p}}}(\partial_p g'_{a\bar{a}})\sqrt{g'^{p\bar{p}}}(\partial_{\bar{p}} g'_{b\bar{b}}) \\ &\leqslant \sum_{a,b}\left(\sum_p g'^{p\bar{p}}|\partial_p g'_{a\bar{a}}|^2\right)^{1/2}\left(\sum_q g'^{q\bar{q}}|\partial_q g'_{b\bar{b}}|^2\right)^{1/2} \\ &= \left(\sum_a \left(\sum_p g'^{p\bar{p}}|\partial_p g'_{a\bar{a}}|^2\right)^{1/2}\right)^2 \\ &= \left(\sum_a \sqrt{g'_{a\bar{a}}}\left(\sum_p g'^{p\bar{p}} g'^{a\bar{a}}|\partial_p g'_{a\bar{a}}|^2\right)^{1/2}\right)^2 \\ &\leqslant \left(\sum_a g'_{a\bar{a}}\right)\left(\sum_{b,p} g'^{p\bar{p}} g'^{b\bar{b}}|\partial_p g'_{b\bar{b}}|^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{(\mathrm{tr}_g g')^2}\sum_{p,a,b} g'^{p\bar{p}}(\partial_p g'_{a\bar{a}})(\partial_{\bar{p}} g'_{b\bar{b}}) &\leqslant \frac{1}{\mathrm{tr}_g g'}\sum_{a,p} g'^{p\bar{p}} g'^{a\bar{a}}|\partial_p g'_{a\bar{a}}|^2 \\ &\leqslant \frac{1}{\mathrm{tr}_g g'}\sum_{a,j,p} g'^{p\bar{p}} g'^{a\bar{a}}|\partial_p g'_{j\bar{a}}|^2,\end{aligned}$$

since in the last sum we are simply adding in some non-negative terms. Finally using the Kähler condition $\partial_p g'_{j\bar{a}} = \partial_j g'_{p\bar{a}}$ we obtain the required inequality from (30). □

Lemma 3.8. *There is a constant C depending on M, ω, $\sup_M |F|$ and a lower bound for ΔF, such that a solution φ of (28) satisfies*

$$C^{-1}(g_{j\bar{k}}) < (g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi) < C(g_{j\bar{k}}).$$

Proof. Using the notation $g'_{j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$ as before, Equation (28) implies

$$-R'_{j\bar{k}} = \partial_j\partial_{\bar{k}}F + \partial_j\partial_{\bar{k}}\varphi - R_{j\bar{k}} = \partial_j\partial_{\bar{k}}F + g'_{j\bar{k}} - g_{j\bar{k}} - R_{j\bar{k}}. \qquad (31)$$

Using Lemma 3.7 we get

$$\Delta' \log \mathrm{tr}_g g' \geqslant -B\mathrm{tr}_{g'} g + \frac{\Delta F + \mathrm{tr}_g g' - n - R}{\mathrm{tr}_g g'},$$

where R is the scalar curvature of g. The Cauchy-Schwarz inequality implies that

$$(\mathrm{tr}_g g')(\mathrm{tr}_{g'} g) \geqslant n^2,$$

and since we are assuming a bound from below on ΔF, we have a constant C such that

$$\Delta' \log \mathrm{tr}_g g' \geqslant -B \mathrm{tr}_{g'} g - C \mathrm{tr}_{g'} g.$$

Now observe that

$$\Delta' \varphi = g'^{j\bar{k}} \partial_j \partial_{\bar{k}} \varphi = g'^{j\bar{k}} (g'_{j\bar{k}} - g_{j\bar{k}}) = n - \mathrm{tr}_{g'} g.$$

It follows that for $A = B + C + 1$ we have

$$\Delta' (\log \mathrm{tr}_g g' - A\varphi) \geqslant \mathrm{tr}_{g'} g - An.$$

Now suppose that $\log \mathrm{tr}_g g' - A\varphi$ achieves its maximum at $p \in M$. Then

$$0 \geqslant \Delta' (\log \mathrm{tr}_g g' - A\varphi)(p) \geqslant \mathrm{tr}_{g'} g(p) - An,$$

so

$$\mathrm{tr}_{g'} g(p) \leqslant An. \tag{32}$$

Choose normal coordinates for g at p, such that g' is diagonal at p. Then (32) implies that at p we have

$$\frac{1}{g'_{i\bar{i}}} = g'^{i\bar{i}} \leqslant An \tag{33}$$

for each i. But from Equation (28) we know that at p

$$\prod_{i=1}^{n} g'_{i\bar{i}} = e^{F(p)+\varphi(p)} \leqslant C_1, \tag{34}$$

for some constant C_1 since we are assuming a bound on $\sup |F|$, from which Lemma 3.6 implies a bound on $\sup |\varphi|$. Now (33) and (34) imply that $g'_{i\bar{i}} \leqslant C_2$ for each i, for some constant C_2. In particular

$$\mathrm{tr}_g g'(p) \leqslant n C_2.$$

Since $\log \mathrm{tr}_g g' - A\varphi$ achieves its maximum at p, we have

$$\log \mathrm{tr}_g g'(x) - A\varphi(x) \leqslant \log \mathrm{tr}_g g'(p) - A\varphi(p) \leqslant \log(nC_2) - A\varphi(p)$$

for any $x \in M$, so since from Lemma 3.6 we can bound $\sup|\varphi|$, we have

$$\sup_M \log \mathrm{tr}_g g' \leqslant C_3$$

for some constant C_3. Now if at a point x we choose normal coordinates for g in which g' is diagonal, then we have an upper bound on $g'_{i\bar{i}}(x)$ for each i. The inequality (34) holds at x too, so we also obtain a lower bound on each $g'_{i\bar{i}}(x)$. These upper and lower bounds on the metric g' are exactly what we wanted to prove. □

3.3 The C^3 and higher order estimates

In this section we will derive estimates for the third derivatives of φ satisfying the equation (28). We will follow the calculation in Phong-Sesum-Sturm [39] which is a more streamlined version of the original proofs in [55], [5], or rather their parabolic analog. It is also possible to use more general techniques to obtain a $C^{2,\alpha}$-estimate given the estimate on $\partial_j\partial_{\bar{k}}\varphi$ in the previous section, namely the complex version of Evans-Krylov's theorem (see [7] or [44] for this approach).

It will be convenient to change our notation slightly. We will write $\hat{g}_{j\bar{k}}$ for the fixed background metric, and $g_{j\bar{k}} = \hat{g}_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$. We will use the equation for the Ricci curvature (31), which we will simply write in the form

$$R_{j\bar{k}} = -g_{j\bar{k}} + T_{j\bar{k}}, \tag{35}$$

where $R_{j\bar{k}}$ is the Ricci curvature of the (unknown) metric g, and $T_{j\bar{k}}$ is a fixed tensor. We will use the estimate from Lemma 3.8, so we know that there is a constant Λ such that

$$\Lambda^{-1}(\hat{g}_{j\bar{k}}) < (g_{j\bar{k}}) < \Lambda(\hat{g}_{j\bar{k}}). \tag{36}$$

We would like to estimate the mixed third derivatives of φ. Since we have already bounded the metric, it is equivalent to estimate the Christoffel symbols $\Gamma^i_{jk} = g^{i\bar{l}}\partial_j g_{k\bar{l}}$. It is more natural to work with tensors, however, so we will focus on the difference of Christoffel symbols

$$S^i_{jk} = \Gamma^i_{jk} - \widehat{\Gamma}^i_{jk}, \tag{37}$$

where $\widehat{\Gamma}^i_{jk}$ are the Christoffel symbols of the Levi-Civita connection of $\hat{g}$. The key calculation now is the following.

Lemma 3.9. *Suppose that g satisfies Equation (35), and the bound (36). There is a constant C depending on M, T, $\hat{g}$ and Λ, such that*

$$\Delta|S|^2 \geqslant -C|S|^2 - C,$$

where $|S|$ is the norm of the tensor S measured with the metric g, and Δ is the g-Laplacian.

Proof. To simplify the notation we will suppress the metric g. We will be computing with the Levi-Civita connection of g, so this will not cause any problems. For instance we will write

$$|S|^2 = g^{j\bar{k}} g^{a\bar{b}} g_{p\bar{q}} S^p_{ja} \overline{S^q_{kb}} = S^p_{ja} \overline{S^p_{ja}},$$

where we are still summing over repeated indices (alternatively we are working at a point in coordinates such that g is the identity). We have

$$\begin{aligned} \Delta|S|^2 &= \nabla_p \nabla_{\bar{p}} (S^i_{jk} \overline{S^i_{jk}}) \\ &= (\nabla_p \nabla_{\bar{p}} S^i_{jk}) \overline{S^i_{jk}} + S^i_{jk} (\overline{\nabla_{\bar{p}} \nabla_p S^i_{jk}}) \\ &\quad + (\nabla_p S^i_{jk})(\overline{\nabla_p S^i_{jk}}) + (\nabla_{\bar{p}} S^i_{jk})(\overline{\nabla_{\bar{p}} S^i_{jk}}) \\ &\geqslant (\nabla_p \nabla_{\bar{p}} S^i_{jk}) \overline{S^i_{jk}} + S^i_{jk} (\overline{\nabla_{\bar{p}} \nabla_p S^i_{jk}}), \end{aligned} \tag{38}$$

since the last two terms in the second line are squares. Commuting derivatives, we have

$$\begin{aligned} \nabla_{\bar{p}} \nabla_p S^i_{jk} &= \nabla_p \nabla_{\bar{p}} S^i_{jk} + R_j{}^m{}_{p\bar{p}} S^i_{mk} + R_k{}^m{}_{p\bar{p}} S^i_{jm} - R_m{}^i{}_{p\bar{p}} S^m_{jk} \\ &= \nabla_p \nabla_{\bar{p}} S^i_{jk} + R_j{}^m S^i_{mk} + R_k{}^m S^i_{jm} - R_m^{\ i} S^m_{jk}, \end{aligned}$$

where $R_j{}^m = g^{m\bar{k}} R_{j\bar{k}}$ is the Ricci tensor of g with an index raised. By Equation (35) and our assumptions, the Ricci tensor is bounded, so

$$|\nabla_{\bar{p}} \nabla_p S^i_{jk}| \leqslant |\nabla_p \nabla_{\bar{p}} S^i_{jk}| + C_1 |S|, \tag{39}$$

for some constant C_1. We also have

$$\begin{aligned} \nabla_p \nabla_{\bar{p}} S^i_{jk} &= \nabla_p \partial_{\bar{p}} (\Gamma^i_{jk} - \widehat{\Gamma}^i_{jk}) \\ &= -\nabla_p (R_j{}^i{}_{k\bar{p}} - \widehat{R}_j{}^i{}_{k\bar{p}}) \\ &= -\nabla_k R_j{}^i + \widehat{\nabla}_p \widehat{R}_j{}^i{}_{k\bar{p}} + (\nabla_p - \widehat{\nabla}_p) \widehat{R}_j{}^i{}_{k\bar{p}}, \end{aligned}$$

where we used the Bianchi identity $\nabla_p R_j{}^i{}_{k\bar{p}} = \nabla_k R_j{}^i{}_{p\bar{p}} = \nabla_k R_j{}^i$, and $\widehat{\nabla}, \widehat{R}$ are the Levi-Civita connection and curvature tensor of $\hat{g}$. The difference in the connections $\nabla_p - \widehat{\nabla}_p$ is bounded by S from the definition (37), and so we can bound the covariant derivative $\nabla_k R^i_j$ using Equation (35). We get

$$|\nabla_p \nabla_{\bar{p}} S^i_{jk}| \leqslant C_2|S| + C_3,$$

for some constants C_2, C_3. Combining this with (39) and (38) we get

$$\Delta|S|^2 \geqslant -(C_4|S| + C_5)|S| = -C_4|S|^2 - C_5|S|,$$

from which the required result follows. □

We are now ready to prove the third order estimate.

Lemma 3.10. *Suppose that g satisfies Equation (35), and the bound (36). Then there is a constant C depending on $M, T, \hat{g}$ and Λ such that $|S| \leqslant C$.*

Proof. Equation (29) from our earlier calculation now implies (in our changed notation) that

$$\Delta \mathrm{tr}_{\hat{g}} g \geqslant -C_1 + \varepsilon|S|^2,$$

for some constants $\varepsilon, C_1 > 0$, since we are assuming that g and $\hat{g}$ are uniformly equivalent. Using the previous lemma, we can then choose a large constant A, such that

$$\Delta(|S|^2 + A\mathrm{tr}_{\hat{g}} g) \geqslant |S|^2 - C_2,$$

for some C_2. Suppose now that $|S|^2 + A\mathrm{tr}_{\hat{g}} g$ achieves its maximum at $p \in M$. Then

$$0 \geqslant |S|^2(p) - C_2,$$

so $|S|^2(p) \leqslant C_2$. Then at every other point $x \in M$ we have

$$|S|^2(x) \leqslant |S|^2(x) + A\mathrm{tr}_{\hat{g}} g(x) \leqslant |S|^2(p) + A\mathrm{tr}_{\hat{g}} g(p) \leqslant C_2 + C_3,$$

for some C_3, which is what we wanted to prove. □

We can finally prove Proposition 3.4, which completes the proof of Aubin-Yau's Theorem 3.1. We recall the statement.

Proposition 3.11. *There exists a constant $C > 0$ depending only on M, ω_0 and F (in the application to Theorem 3.1 F is computed from ω_0), such that if φ_t satisfies the equation*

$$(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = e^{tF+\varphi_t}\omega_0^n$$

for some $t \in [0,1]$, then

$$(g_{j\bar{k}} + \partial_j \partial_{\bar{k}} \varphi_t) > C^{-1}(g_{j\bar{k}}),$$

and

$$\|\varphi_t\|_{C^{3,\alpha}(M)} \leqslant C,$$

where the Hölder norm is measured with respect to the metric ω_0.

Proof. Lemmas 3.6 and 3.8 together show that $g_{j\bar{k}} + \partial_j \partial_{\bar{k}} \varphi$ is uniformly equivalent to $g_{j\bar{k}}$. Then 3.10 shows that we have an a priori bound on the mixed third derivatives $\partial_j \partial_{\bar{k}} \partial_l \varphi$, and $\partial_{\bar{j}} \partial_{\bar{k}} \partial_l \varphi$. In particular this gives C^α bounds on $\partial_j \partial_{\bar{k}} \varphi$. Now we can use the same argument of differentiating the equation and using the Schauder estimates as in Lemma 3.3 to get an a priori bound on $\|\varphi\|_{C^{3,\alpha}}$. □

3.4 Discussion of the $c_1(M) = 0$ and $c_1(M) > 0$ cases

In view of Theorem 3.1 it is natural to ask what happens when $c_1(M) = 0$ or $c_1(M) > 0$. Let us suppose first that $c_1(M) = 0$. In this case we are looking for Ricci flat metrics. Given any Kähler metric ω on M (we are still taking M to be compact), the Ricci form of ω is exact, so by the $\partial\bar{\partial}$-lemma there is a function F such that

$$\mathrm{Ric}(\omega) = \sqrt{-1}\partial\bar{\partial}F.$$

Arguing as in the beginning of Section 3.1, we see that for $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ to be Ricci flat, we need to solve the equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n.$$

A slight difference from before is that for this to be possible, we first need to normalize F by adding a constant. In fact by integrating both sides of the equation, we have

$$\int_M e^F \omega^n = \int_M (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \int_M \omega^n,$$

where we used that

$$\begin{aligned}(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n - \omega^n &= \sqrt{-1}\partial\bar{\partial}\varphi \wedge (\omega'^{n-1} + \omega'^{n-2} \wedge \omega + \ldots + \omega^{n-1}) \\ &= d(\sqrt{-1}\bar{\partial}\varphi \wedge (\omega'^{n-1} + \ldots + \omega^{n-1}))\end{aligned}$$

is exact, so the volume of M with respect to the two different metrics is equal. The following theorem completely answers the $c_1(M) = 0$ case.

Theorem 3.12 (Yau). *Let (M, ω) be a compact Kähler manifold, and $F : M \to \mathbf{R}$ a smooth function such that*

$$\int_M e^F \omega^n = \int_M \omega^n.$$

Then there is a smooth function $\varphi : M \to \mathbf{R}$, unique up to the addition of a constant, such that $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is a positive form, and

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F \omega^n.$$

The equation looks very similar to Equation 26 that we had to solve when proving Theorem 3.1, but there is one crucial difference. It is now not possible to prove an a priori estimate for $\sup_M |\varphi|$ using the maximum principle like we did in Lemma 3.6, since the function φ does not appear on the right hand side of the equation. Nevertheless one can estimate $\sup_M |\varphi|$ using more sophisticated arguments, due to Yau [55]. For the details see any of the references [55], [7], [44] or [52]. Once we have an estimate for $\sup_M |\varphi|$, we can obtain higher order estimates in exactly the same way as was done in Lemmas 3.8 and 3.10. The "openness" argument of Lemma 3.3 also goes through with minor changes, so the equation can be solved using the continuity method. For the uniqueness statement see Exercise 3.6. We should also mention that Theorem 1.19 is a simple consequence of Theorem 3.12.

The remaining case is when $c_1(M) > 0$. Suppose that $\omega \in 2\pi c_1(M)$ is any Kähler metric. We are now seeking a metric $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ such that $\mathrm{Ric}(\omega') = \omega'$. Arguing just like at the beginning of Section 3.1 this requires solving the equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F-\varphi}\omega^n.$$

Again, this equation looks very similar to the one we already solved, but once again we cannot use the maximum principle to obtain an estimate for $\sup_M |\varphi|$ because the sign of φ is reversed. If we had such an estimate, then the same arguments as before could be used to solve the equation. It turns out, however, that not every manifold with $c_1(M) > 0$ admits a Kähler-Einstein metric, so in fact the equation can not always be solved. The first obstructions due to Matsushima [34] and Futaki [20] were based on the automorphism group of M, and in the case of complex surfaces these turned out to be sufficient by the work of Tian [50]. Later a much more subtle obstruction called K-stability was found by Tian [51] motivated by a conjecture due to Yau [56]. In the remainder of this course we will study these obstructions, in particular K-stability. Instead of Kähler-Einstein metrics it is most natural to study a larger class of metrics introduced by Calabi [8], called extremal metrics. It is these metrics that we will start to study in the next section.

3.5 Exercises

Exercise 3.1. Suppose that L is a line bundle over a complex manifold M, and s is a global holomorphic section of L such that the zero set $s^{-1}(0)$ is a smooth submanifold $D \subset M$. The normal bundle N_D of D in M is defined to be the quotient bundle $(TM|_D)/TD$, where $TM|_D$ is the restriction of the holomorphic tangent bundle of M to D. Show that

$$N_D = L|_D,$$

where $L|_D$ is the restriction of L to D.

Exercise 3.2. Let M be a complex manifold, and suppose that $D \subset M$ is a complex submanifold with (complex) codimension 1. Show that the canonical bundles of D and M are related by

$$K_D = (K_M|_D) \otimes N_D,$$

where $K_M|_D$ is the restriction of the canonical bundle of M to D. This is called the adjunction formula.

Exercise 3.3. Suppose that $M \subset \mathbf{P}^n$ is a smooth hypersurface of degree d, i.e. M is defined by the vanishing of a section of $\mathcal{O}(d)$. Show that if $d > n+1$, then $c_1(M) < 0$, so in particular M admits a Kähler-Einstein metric.

More generally, suppose that $M \subset \mathbf{P}^n$ is a smooth complex submanifold of codimension r, defined by the intersection of r hypersurfaces of degrees $d_1, \ldots, d_r$. If $d_1 + \ldots + d_r > n+1$, then show that $c_1(M) < 0$.

Exercise 3.4. Under the same assumptions as Lemma 3.9 show that there is a constant C such that

$$\Delta|\mathrm{Rm}|^2 \geqslant -C|\mathrm{Rm}|^3 - C|\mathrm{Rm}| + |\nabla\mathrm{Rm}|^2 + |\overline{\nabla}\mathrm{Rm}|^2,$$

where Rm is the curvature tensor of g, so

$$|\mathrm{Rm}|^2 = g^{i\bar{p}}g^{k\bar{r}}g^{s\bar{l}}g_{j\bar{q}}R_{i\,k\bar{l}}^{\,j}\overline{R_{p\,r\bar{s}}^{\,q}},$$

and $\nabla \mathrm{Rm} = \nabla_p R_{i\ k\bar{l}}^{\ j}$ and $\overline{\nabla} \mathrm{Rm} = \nabla_{\bar{p}} R_{i\ k\bar{l}}^{\ j}$. Using this, show that there is a constant C such that

$$\Delta|\mathrm{Rm}| \geqslant -C|\mathrm{Rm}|^2 - C,$$

and finally using an argument similar to Lemma 3.10 show that under the same assumptions $|\mathrm{Rm}| \leqslant C$ for some C.

Exercise 3.5. Generalize the previous exercise to higher order derivatives of the curvature, $|\nabla^k \mathrm{Rm}|$. In this way one can obtain a priori bounds on higher derivatives of a solution φ of the Equation 26, without appealing to the Schauder estimates.

Exercise 3.6. Suppose that (M, ω) is a compact Kähler manifold, and $\varphi : M \to \mathbf{R}$ is such that

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = \omega^n.$$

Show that φ must be a constant, by using the identity

$$\int_M \varphi[(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n - \omega^n] = 0,$$

and integrating by parts. This proves the uniqueness statement in Theorem 3.12. A useful equality is

$$\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^{n-1} = \frac{1}{n}|\partial\varphi|^2\omega^n,$$

which is a consequence of Lemma 4.6 in the next section.

4 Extremal metrics

Suppose that M is a compact Kähler manifold, with a Kähler class $\Omega \in H^2(M, \mathbf{R})$. A natural question is to ask for a particularly nice metric representing the class Ω. In the previous section we have seen that if $c_1(M) < 0$ and $\Omega = -c_1(M)$, then M admits a unique Kähler-Einstein metric, while if $c_1(M) = 0$, then any Kähler class on M admits a unique Ricci flat metric. Extermal metrics, introduced by Calabi [8], are a natural generalization of these to arbitrary Kähler classes on compact Kähler manifolds. When they exist, extremal metrics are good candidates for being the "best" metrics in a given Kähler class. In this section we will introduce extremal metrics and study some of their basic properties, while later in the course we will study obstructions to their existence.

4.1 The Calabi functional

As above, suppose that M is a compact Kähler manifold and $\Omega \in H^2(M, \mathbf{R})$ is a Kähler class.

Definition 4.1. An extremal metric on M in the class Ω is a critical point of the functional

$$\mathrm{Cal}(\omega) = \int_M S(\omega)^2 \, \omega^n,$$

for $\omega \in \Omega$, where $S(\omega)$ is the scalar curvature. This functional is called the Calabi functional.

The first important result is understanding the Euler-Lagrange equation characterizing extremal metrics. For a function $f : M \to \mathbf{R}$ on a Kähler manifold, let us write $\mathrm{grad}^{1,0} f = g^{j\bar{k}} \partial_{\bar{k}} f$. This is a section of $T^{1,0}M$, and it is (up to a factor of 4), the (1,0)-part of the Riemannian gradient of f.

Theorem 4.2. *A metric ω on M is extremal if and only if* $\mathrm{grad}^{1,0} S(\omega)$ *is a holomorphic vector field.*

Proof. First let us study the variation of the Calabi functional under variations of a Kähler metric in a fixed Kähler class. So let $\omega_t = \omega + t\sqrt{-1}\partial\bar{\partial}\varphi$, and we will compute the derivative of $\mathrm{Cal}(\omega_t)$ at $t = 0$. We have

$$\left.\frac{d}{dt}\right|_{t=0} \omega_t^n = n\sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega^{n-1} = \Delta\varphi \, \omega^n,$$

and so

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ric}(\omega_t) = -\sqrt{-1}\partial\bar{\partial}\Delta\varphi.$$

Using that $S(\omega_t) = g_t^{j\bar{k}} R_{t,j\bar{k}}$,where $R_{t,j\bar{k}}$ is the Ricci curvature of ω_t, we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} S(\omega_t) &= -g^{j\bar{q}}(\partial_p\partial_{\bar{q}}\varphi)g^{p\bar{k}}R_{j\bar{k}} - \Delta^2\varphi \\ &= -\Delta^2\varphi - R^{\bar{k}j}\partial_j\partial_{\bar{k}}\varphi.\end{aligned}$$

Writing $S = S(\omega)$ for simplicity, it follows that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \mathrm{Cal}(\omega) &= \int_M [-2S(\Delta^2\varphi + R^{\bar{k}j}\partial_j\partial_{\bar{k}}\varphi) + S^2\Delta\varphi]\omega^n \\ &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j\nabla_{\bar{k}}(R^{\bar{k}j}S) + \Delta(S^2)\big]\omega^n\end{aligned}$$

Using the Bianchi identity $\nabla_{\bar{k}}R^{\bar{k}j} = g^{j\bar{k}}\nabla_{\bar{k}}S$, we have

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \mathrm{Cal}(\omega) &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j(Sg^{j\bar{k}}\nabla_{\bar{k}}S + R^{\bar{k}j}\nabla_{\bar{k}}S) + \Delta(S^2)\big]\omega^n \\ &= \int_M \varphi\big[-2\Delta^2 S - 2\nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}S)\big]\omega^n.\end{aligned}$$

In particular if ω is an extremal metric, then this variation must vanish for every φ, so

$$\Delta^2 S + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}S) = 0.$$

Commuting derivatives, for any function ψ we have

$$\begin{aligned}\Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_{\bar{k}}\nabla_p\nabla_{\bar{q}}\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) \\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_j\nabla_p\nabla_{\bar{k}}\nabla_{\bar{q}}\psi - g^{j\bar{k}}g^{p\bar{q}}\nabla_j(R^{\bar{m}}_{\bar{q}p\bar{k}}\nabla_{\bar{m}}\psi) \\ &\qquad + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) \\ &= g^{j\bar{k}}g^{p\bar{q}}\nabla_p\nabla_j\nabla_{\bar{k}}\nabla_{\bar{q}}\psi.\end{aligned}$$

It follows that if we write

$$\begin{aligned}\mathcal{D} : C^\infty(M, \mathbf{C}) &\to C^\infty(\Omega^{0,1}M \otimes \Omega^{0,1}M) \\ \psi &\mapsto \nabla_{\bar{k}}\nabla_{\bar{q}}\psi,\end{aligned}$$

then

$$\Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi) = \mathcal{D}^*\mathcal{D}\psi,$$

where $\mathcal{D}^*$ is the formal adjoint of $\mathcal{D}$. In particular if $\mathcal{D}^*\mathcal{D}S = 0$, then

$$0 = \int_M S\mathcal{D}^*\mathcal{D}S\,\omega^n = \int_M |\mathcal{D}S|^2\,\omega^n,$$

so $\mathcal{D}S = 0$. Using the metric to identify $\Omega^{0,1}M \cong T^{1,0}M$, the operator $\mathcal{D}$ can also be thought of as

$$\mathcal{D}(\psi) = \nabla_{\bar{k}}(g^{j\bar{q}}\nabla_{\bar{q}}\psi) = \nabla_{\bar{k}}(\mathrm{grad}^{1,0}\psi) = \overline{\partial}(\mathrm{grad}^{1,0}\psi),$$

since on the holomorphic tangent bundle $T^{1,0}M$ the $(0,1)$-part of the covariant derivative coincides with the usual antiholomorphic partial derivatives. Therefore $\mathcal{D}S = 0$ is equivalent to saying that $\mathrm{grad}^{1,0}S$ is holomorphic. $\square$

Definition 4.3. The 4th order operator that appeared in the previous proof:

$$\begin{aligned}\mathcal{D}^*\mathcal{D}\psi &= \Delta^2\psi + \nabla_j(R^{\bar{k}j}\nabla_{\bar{k}}\psi)\\ &= \Delta^2\psi + R^{\bar{k}j}\nabla_j\nabla_{\bar{k}}\psi + g^{j\bar{k}}\nabla_j S\nabla_{\bar{k}}\psi,\end{aligned}$$

is called the Lichnerowicz operator. We saw in the proof that on a compact Kähler manifold $\mathcal{D}^*\mathcal{D}\psi = 0$ if and only if $\mathrm{grad}^{1,0}\psi$ is holomorphic. Note that in general this is a complex operator, unless S is constant. One must remember this when using the self-adjointness of $\mathcal{D}^*\mathcal{D}$. For instance for complex valued functions f, g we have

$$\int_M (\mathcal{D}^*\mathcal{D}f)\bar{g}\,\omega^n = \int_M f\overline{\mathcal{D}^*\mathcal{D}g}\,\omega^n.$$

From the previous proof we obtain a useful description of the variation of the scalar curvature, under a variation of the metric.

Lemma 4.4. *Suppose that $\omega_t = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi$. Then the scalar curvature S_t of ω_t satisfies*

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} S_t &= -\mathcal{D}^*\mathcal{D}\varphi + g^{j\bar{k}}\nabla_j S\nabla_{\bar{k}}\varphi\\ &= -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\bar{k}}\nabla_j\varphi\nabla_{\bar{k}}S.\end{aligned}$$

Proof. The first formula follows from the previous proof. The second one follows by taking the conjugate, and noting that S_t is real. $\square$

Example 4.5. The most important examples of extremal metrics are constant scalar curvature Kähler metrics, which we will abbreviate as cscK. In fact most compact Kähler manifolds admit no non-zero holomorphic vector fields at all, so on such manifolds an extremal metric necessarily has constant scalar curvature.

In particular Kähler-Einstein metrics have constant scalar curvature, so they are examples of extremal metrics. Conversely suppose that ω is a cscK metric, and we are in a Kähler class where a Kähler-Einstein metric could exist, i.e. $c_1(M) = \lambda[\omega]$ for some λ. Then ω is in fact Kähler-Einstein. Indeed, if the scalar curvature S is constant, then

$$\overline{\partial}^* R_{j\bar{k}} = -g^{p\bar{k}}\nabla_p R_{j\bar{k}} = -\nabla_j S = 0,$$

so the Ricci form is harmonic. But $2\pi\lambda\omega$ is also a harmonic form in the same class, so we have $R_{j\bar{k}} = 2\pi\lambda g_{j\bar{k}}$.

We will see in Section 4.4 that there are also examples of extremal metrics which do not have constant scalar curvature.

In the next section we will study the interplay between holomorphic vector fields and extremal metrics further. In the remainder of this section we will show that in the definition of extremal metrics, instead of taking the L^2-norm of the scalar curvature, we could equivalently have taken the L^2-norms of the Ricci, or Riemannian curvatures. For this we first need the following.

Lemma 4.6. *Let α and β be (1,1)-forms, given in local coordinates by $\alpha = \sqrt{-1}\alpha_{j\bar{k}}dz^j \wedge d\bar{z}^k$ and $\beta = \sqrt{-1}\beta_{j\bar{k}}dz^j \wedge d\bar{z}^k$, such that $\alpha_{j\bar{k}}$ and $\beta_{j\bar{k}}$ are Hermitian matrices. If ω is a Kähler metric with components $g_{j\bar{k}}$, then*

$$\begin{aligned} n\alpha\wedge\omega^{n-1} &= (\mathrm{tr}_\omega\alpha)\omega^n \\ n(n-1)\alpha\wedge\beta\wedge\omega^{n-2} &= \big[(\mathrm{tr}_\omega\alpha)(\mathrm{tr}_\omega\beta) - \langle\alpha,\beta\rangle_\omega\big]\omega^n, \end{aligned}$$

where $\mathrm{tr}_\omega\alpha = g^{j\bar{k}}\alpha_{j\bar{k}}$ and $\langle\alpha,\beta\rangle_\omega = g^{j\bar{k}}g^{p\bar{q}}\alpha_{j\bar{q}}\beta_{p\bar{k}}$.

Proof. We will prove the second equality since the first follows by taking $\beta = \omega$. We compute in local coordinates at a point where g is the identity, and α is diagonal. Then

$$\omega = \sqrt{-1}\sum_i g_{i\bar{i}}dz^i\wedge d\bar{z}^i,$$

so

$$\begin{aligned} \omega^{n-2} = (\sqrt{-1})^{n-2}(n-2)!\sum_{i<j} dz^1\wedge d\bar{z}^1\wedge\ldots\wedge \widehat{dz^i\wedge d\bar{z}^i}\wedge\ldots \\ \wedge\,\widehat{dz^j\wedge d\bar{z}^j}\wedge\ldots\wedge dz^n\wedge d\bar{z}^n, \end{aligned}$$

where the hats mean that those terms are omitted. Also

$$\begin{aligned} \alpha\wedge\beta = (\sqrt{-1})^2\sum_{i\neq j}\alpha_{i\bar{i}}\beta_{j\bar{j}}dz^i\wedge d\bar{z}^i\wedge dz^j\wedge d\bar{z}^j \\ + (\text{terms involving } \beta_{j\bar{k}} \text{ with } j\neq k), \end{aligned}$$

since α is diagonal. It follows that

$$\begin{aligned}
n(n-1)\alpha \wedge \beta \wedge \omega^{n-2} &= (\sqrt{-1})^n n! \sum_{i\neq j} \alpha_{i\bar{i}}\beta_{j\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n \\
&= \left(\sum_{i\neq j} \alpha_{i\bar{i}}\beta_{j\bar{j}} \right) \omega^n \\
&= \left(\sum_{i,j} \alpha_{i\bar{i}}\beta_{j\bar{j}} - \sum_{i} \alpha_{i\bar{i}}\beta_{i\bar{i}} \right) \omega^n \\
&= \left[(\mathrm{tr}_\omega \alpha)(\mathrm{tr}_\omega \beta) - \langle \alpha, \beta \rangle_\omega \right] \omega^n.
\end{aligned}$$

□

We can now compare the different functionals obtained by taking the L^2-norms of the Ricci and Riemannian curvatures.

Corollary 4.7. *There are constants C_1, C_2 depending on M and the Kähler class Ω, such that if $\omega \in \Omega$, then*

$$\begin{aligned}
\int_M S\,\omega^n &= 2n\pi c_1(M) \cup [\omega]^{n-1}, \\
\int_M |\mathrm{Ric}|^2\,\omega^n &= \int_M S^2\,\omega^n + C_1, \\
\int_M |\mathrm{Rm}|^2\,\omega^n &= \int_M |\mathrm{Ric}|^2\,\omega^n + C_2,
\end{aligned}$$

where S, Ric *and* Rm *are the scalar, Ricci, and Riemannian curvatures of ω.*

Proof. Let us write $\rho = \sqrt{-1} R_{j\bar{k}} dz^j \wedge d\bar{z}^k$ for the Ricci form of ω, and $g_{j\bar{k}}$ for the local components of the metric ω. Applying the previous lemma, we have

$$\int_M S\,\omega^n = n \int_M \rho \wedge \omega^{n-1} = 2n\pi c_1(M) \cup [\omega]^{n-1},$$

since $\mathrm{tr}_\omega \rho = S$, and ρ is a closed form representing the cohomology class $2\pi c_1(M)$.

For the second identity we again apply the previous lemma.

$$\int_M (S^2 - |\mathrm{Ric}|^2)\omega^n = n(n-1) \int_M \rho \wedge \rho \wedge \omega^{n-2} = 4n(n-1)\pi^2 c_1(M)^2 \cup [\omega]^{n-2},$$

since $\langle \rho, \rho \rangle_\omega = |\mathrm{Ric}|^2$.

For the third equation, let us introduce the endomorphism valued 2-form $\Theta_p^{\ q}$ defined by

$$\Theta_p^{\ q} = \sqrt{-1} R_{p\ j\bar{k}}^{\ q} dz^j \wedge d\bar{z}^k.$$

Applying the previous lemma we have

$$\begin{aligned} n(n-1)\Theta_p^{\ q} \wedge \Theta_q^{\ p} \wedge \omega^{n-2} &= \left(R_p^{\ q} R_q^{\ p} - g^{j\bar{b}} g^{a\bar{k}} R_{p\ j\bar{k}}^{\ q} R_{q\ a\bar{b}}^{\ p} \right) \omega^n \\ &= \left(|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2 \right) \omega^n. \end{aligned}$$

The (2,2)-form $\Theta_p^{\ q} \wedge \Theta_q^{\ p}$ is a closed form whose cohomology class is independent of the metric (in fact it is the characteristic class $4\pi^2 c_1(M)^2 - 8\pi^2 c_2(M)$), and therefore

$$\int_M (|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2)\, \omega^n = C_2.$$

$\square$

For us the most important point from the previous result is that the average scalar curvature

$$\hat{S} = \frac{2n\pi c_1(M) \cup [\omega]^{n-1}}{[\omega]^n}$$

only depends on M and the Kähler class $[\omega]$. Since

$$\int_M S(\omega)^2 \, \omega^n = \int_M (S(\omega) - \hat{S})^2 \, \omega^n + \int_M \hat{S}^2 \, \omega^n,$$

if a cscK metric exists in a Kähler class, then it minimizes the Calabi functional. It turns out that more generally extremal metrics minimize the Calabi functional in their respective Kähler classes, but this is much harder to prove (see Chen [10] and Donaldson [17]).

Remark 4.8. An important consequence of the previous result is that if ω is an extremal metric, then we have an estimate for the L^2-norm of the curvature of ω. This can be exploited to understand how a family of extremal metrics could degenerate in certain cases. See for example Chen-LeBrun-Weber [11] for an existence result based on a careful analysis of the possible "blow-up" behaviors.

4.2 Holomorphic vector fields and the Futaki invariant

As before, M is a compact Kähler manifold, with Kähler metric ω. A holomorphic vector field is a holomorphic section of $T^{1,0}M$. We will focus our attention on those vector fields, which can be written as $v^j = g^{j\bar{k}}\partial_{\bar{k}}f$ for a function f. It is natural to allow complex valued functions too. Let us define

$$\begin{aligned} \mathfrak{h} := \{&\text{holomorphic sections } v \text{ of } T^{1,0}M, \\ &\text{such that } v^j = g^{j\bar{k}}\partial_{\bar{k}}f \text{ for some } f : M \to \mathbf{C}\}. \end{aligned}$$

The space $\mathfrak{h}$ is independent of the choice of metric in the Kähler class $[\omega]$, because of the following.

Lemma 4.9. *Let us write $g_{\varphi,j\bar{k}} = g_{j\bar{k}} + \partial_j\partial_{\bar{k}}\varphi$ for some φ. If $v \in \mathfrak{h}$ and $v^j = g^{j\bar{k}}\partial_{\bar{k}}f$, then*

$$v^j = g_\varphi^{j\bar{k}}\partial_{\bar{k}}(f + v(\varphi)),$$

where $v(\varphi) = v^i\partial_i\varphi$ is the derivative of φ along v.

Proof. We have

$$g_{\varphi,j\bar{p}}v^j = (g_{j\bar{p}} + \partial_j\partial_{\bar{p}}\varphi)g^{j\bar{k}}\partial_{\bar{k}}f = \partial_{\bar{p}}f + \partial_{\bar{p}}(v^j\partial_j\varphi),$$

where we used that $\nabla_{\bar{p}}v^j = \partial_{\bar{p}}v^j = 0$ since v is holomorphic. Multiplying this equation by the inverse of g_φ we get the required result. □

Given a holomorphic vector field $v \in \mathfrak{h}$ let us call $f : M \to \mathbf{C}$ a holomorphy potential for v if $v^j = g^{j\bar{k}}\partial_{\bar{k}}f$. Holomorphy potentials are unique up to addition of a constant, and for a fixed vector field the previous lemma describes how to vary the holomorphy potential as we vary the metric. It is useful to note that holomorphy potentials are precisely given by the kernel of $\mathcal{D}^*\mathcal{D}$.

Remark 4.10. It turns out that $\mathfrak{h}$ consists of precisely those holomorphic vector fields which have a zero somewhere (see LeBrun-Simanca [26]), so $\mathfrak{h}$ does not even depend on the choice of Kähler class.

Remark 4.11. It is often useful to think of sections of $T^{1,0}M$ as real vector fields. This can be achieved by identifying $T^{1,0}M$ with the real tangent bundle TM, mapping a vector field of type (1,0) to its real part. In local coordinates $z^i = x^i + \sqrt{-1}y^i$, in view of Equation (3), this means that

$$\frac{\partial}{\partial z^i} \mapsto \frac{1}{2}\frac{\partial}{\partial x^i} \qquad \sqrt{-1}\frac{\partial}{\partial z^i} \mapsto \frac{1}{2}\frac{\partial}{\partial y^i}.$$

We can then calculate that if $f = u + \sqrt{-1}v$ is the decomposition of f into its real and imaginary parts, then

$$g^{j\bar{k}}\partial_{\bar{k}} f \mapsto \frac{1}{2}(\mathrm{grad}\, u + J\mathrm{grad}\, v),$$

where grad is the usual Riemannian gradient, and J is the complex structure. We will see in Section 5.1 that $J\mathrm{grad}\, v$ is the Hamiltonian vector field corresponding to v with respect to the symplectic form ω. It follows that if $v \in \mathfrak{h}$ has a purely imaginary holomorphy potential, then the real part of v is a Killing field.

The following theorem, due to Futaki [20] gives an obstruction to finding cscK metrics in a Kähler class. It will turn out to be a first glimpse into the obstruction given by K-stability.

Theorem 4.12. *Let (M, ω) be a compact Kähler manifold. Let us define the functional $F : \mathfrak{h} \to \mathbf{C}$, called the Futaki invariant, by*

$$F(v) = \int_M f(S - \hat{S})\, \omega^n, \tag{40}$$

where f is a holomorphy potential for v, and $\hat{S}$ is the average of the scalar curvature S. This functional is independent of the choice of metric in the Kähler class $[\omega]$. In particular if $[\omega]$ admits a cscK metric, then $F(v) = 0$ for all $v \in \mathfrak{h}$.

Proof. Suppose that $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is another Kähler metric in $[\omega]$, and write $\omega_t = \omega + t\sqrt{-1}\partial\overline{\partial}\varphi$. Let

$$F_t(v) = \int_M f_t(S_t - \hat{S})\omega_t^n,$$

where f_t is a holomorphy potential for v with respect to ω_t, and S_t is the scalar curvature of ω_t. Note that by Corollary 4.7 the average $\hat{S}$ is independent of t. It is enough to show that the derivative of $F_t(v)$ at $t = 0$ vanishes. By Lemma 4.9, we can choose f_t so that

$$\left.\frac{d}{dt}\right|_{t=0} f_t = v^j\partial_j\varphi = g^{j\bar{k}}\partial_{\bar{k}} f\partial_j\varphi,$$

and from the proof of Theorem 4.2 and Lemma 4.4 we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \omega_t^n &= \Delta\varphi\, \omega^n \\ \left.\frac{d}{dt}\right|_{t=0} S_t &= -\overline{\mathcal{D}^*\mathcal{D}\varphi} + g^{j\bar{k}}\partial_j\varphi\partial_{\bar{k}} S.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} F_t(v) &= \int_M \big[g^{j\bar{k}}\partial_{\bar{k}}f\partial_j\varphi(S-\hat{S}) - f(\overline{\mathcal{D}^*\mathcal{D}\varphi} - g^{j\bar{k}}\partial_j\varphi\partial_{\bar{k}}S) \\ &\qquad + f(S-\hat{S})\Delta\varphi\big]\,\omega^n \\ &= \int_M -f\overline{\mathcal{D}^*\mathcal{D}\varphi}\,\omega^n \\ &= -\int_M \varphi\mathcal{D}^*\mathcal{D}f\,\omega^n,\end{aligned}$$

after writing $\Delta\varphi = g^{j\bar{k}}\partial_{\bar{k}}\partial_j\varphi$ and integrating by parts. Using that f is a holomorphy potential, we have $\mathcal{D}^*\mathcal{D}f = 0$, so the result follows. □

To compute the Futaki invariant using the defining formula directly is impractical, if not impossible in all but the simplest cases. Instead, it is possible to use a localization formula to compute $F(v)$ for a holomorphic vector field, by studying the zero set of v (see Tian [52]). A third approach, which will be fundamental in the later developments, is that if M is a projective manifold then the Futaki invariant can be computed algebro-geometrically. We will explain this in Section 6.3.

A useful corollary to the previous theorem is the following.

Corollary 4.13. *Suppose that ω is an extremal metric on a compact Kähler manifold M. If the Futaki invariant vanishes (relative to the Kähler class $[\omega]$), then ω has constant scalar curvature.*

Proof. Since ω is an extremal metric, the vector field $v^j = g^{j\bar{k}}\partial_{\bar{k}}S$ is in $\mathfrak{h}$. It follows that

$$0 = F(v) = \int_M S(S-\hat{S})\omega^n = \int_M (S-\hat{S})^2\,\omega^n,$$

so we must have $S = \hat{S}$, i.e. S is constant. □

4.3 The Mabuchi functional and geodesics

In this section we will see that cscK metrics have an interesting variational characterization, discovered by Mabuchi [32], which is different from being critical points of the Calabi functional. Moreover this variational point of view gives insight into when we can expect a cscK metric to exist.

As before, let (M,ω) be a compact Kähler manifold. Let us write

$$\mathcal{K} = \{\varphi : M \to \mathbf{R} \mid \varphi \text{ is smooth, and } \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0\},$$

for the space of Kähler potentials for Kähler metrics in the class $[\omega]$. For any $\varphi \in \mathcal{K}$ we will write

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

for the corresponding Kähler metric, and we will put a φ subscript on other geometric quantities to indicate that they refer to this metric. The tangent space $T_\varphi\mathcal{K}$ at φ can be identified with the smooth real valued functions $C^\infty(M)$. We can therefore define a 1-form α on $\mathcal{K}$ by letting

$$\alpha_\varphi(\psi) = \int_M \psi(\hat{S} - S_\varphi)\,\omega_\varphi^n.$$

We can check that this 1-form is closed. This boils down to differentiating $\alpha_\varphi(\psi)$ with respect to φ, and showing that the resulting 2-tensor is symmetric. More precisely we need to compute

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1),$$

and show that it is symmetric in ψ_1 and ψ_2. We have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi+t\psi_2}(\psi_1) &= \int_M \left[\psi_1(\mathcal{D}^*_\varphi\mathcal{D}_\varphi\psi_2 - g_\varphi^{j\bar{k}}\partial_j S_\varphi \partial_{\bar{k}}\psi_2) + \psi_1(\hat{S} - S_\varphi)\Delta_\varphi\psi_2\right]\omega_\varphi^n \\ &= \int_M \left[\psi_1\mathcal{D}^*_\varphi\mathcal{D}_\varphi\psi_2 - (\hat{S} - S_\varphi)g^{j\bar{k}}\partial_j\psi_1\partial_{\bar{k}}\psi_2\right]\omega_\varphi^n.\end{aligned}$$

Switching ψ_1 and ψ_2 amounts to taking the conjugate of the whole expression (using self-adjointness of the complex operator $\mathcal{D}^*\mathcal{D}$). The left hand side of the equation is real, so it follows that the expression is symmetric in ψ_1 and ψ_2.

Since α is a closed form and $\mathcal{K}$ is contractible, there exists a function $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ such that $d\mathcal{M} = \alpha$ which we can normalize so that $\mathcal{M}(0) = 0$. We could get a more explicit formula by integrating α along straight lines, but the variation of $\mathcal{M}$ is more important for us. To summarize, we have the following.

Proposition 4.14. *There is a functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$, such that the variation of $\mathcal{M}$ along a path $\varphi_t = \varphi + t\psi$ is given by*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{M}(\varphi_t) = \int_M \psi(\hat{S} - S_\varphi)\omega_\varphi^n, \tag{41}$$

where S_φ is the scalar curvature of the metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$. This is called the Mabuchi functional or the K-energy.

Note that since the variation of $\mathcal{M}$ in the direction of the constant functions vanishes, $\mathcal{M}$ actually descends to a functional on the space of Kähler metrics in $[\omega]$. Moreover it is clear that critical points of $\mathcal{M}$ are given by constant scalar curvature metrics.

Next we will show that $\mathcal{M}$ is a convex function on $\mathcal{K}$, if we endow $\mathcal{K}$ with a natural Riemannian metric, introduced by Mabuchi [33] (see also Semmes [41] and Donaldson [15]). Given two elements $\psi_1, \psi_2 \in T_\varphi\mathcal{K}$ in the tangent space at $\varphi \in \mathcal{K}$, we can define the inner product

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1 \psi_2 \, \omega_\varphi^n.$$

This defines a Riemannian metric on the infinite dimensional space $\mathcal{K}$. Let us first compute the equation satisfied by geodesics.

Proposition 4.15. *A path $\varphi_t \in \mathcal{K}$ is a (constant speed) geodesic if and only if*

$$\ddot{\varphi}_t - |\partial\dot{\varphi}_t|_t^2 = \ddot{\varphi}_t - g_t^{j\bar{k}}\partial_j\dot{\varphi}_t\partial_{\bar{k}}\dot{\varphi}_t = 0,$$

where the dots mean t-derivatives, and g_t is the metric $\omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$.

Proof. A constant speed geodesic is a critical point of the energy of a path. The energy of the path φ_t for $t \in [0,1]$, say, is

$$E(\varphi_t) = \int_0^1 \int_M \dot{\varphi}_t^2 \, \omega_t^n \, dt.$$

Under a variation $\varphi_t + \varepsilon\psi_t$, where ψ_t vanishes at $t = 0$ and $t = 1$ we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} E(\varphi_t + \varepsilon\psi_t) &= \int_0^1 \int_M (2\dot{\varphi}_t\dot{\psi}_t + \dot{\varphi}_t^2\Delta_t\psi_t)\, \omega_t^n \, dt \\
&= \int_0^1 \int_M (2\dot{\varphi}_t\dot{\psi}_t + \Delta(\dot{\varphi}_t^2)\psi_t)\, \omega_t^n \, dt \\
&= \int_0^1 \int_M \big[-2\ddot{\varphi}_t\psi_t - 2\dot{\varphi}_t\psi_t\Delta_t\dot{\varphi}_t + \Delta_t(\dot{\varphi}_t^2)\psi_t \big]\, \omega_t^n \, dt \\
&= \int_0^1 \int_M -2\psi_t\big[\ddot{\varphi}_t - g_t^{j\bar{k}}\partial_j\dot{\varphi}_t\partial_{\bar{k}}\dot{\varphi}_t\big]\, \omega_t^n \, dt,
\end{aligned}$$

where we integrated by parts on the manifold, and also with respect to t (the $\Delta_t\dot{\varphi}_t$ term in the third line comes from differentiating ω_t^n with respect to t). The required expression for the geodesic equation follows. $\square$

Example 4.16. A useful family of geodesics arises as follows. Suppose that $v \in \mathfrak{h}$ has holomorphy potential $u : M \to \mathbf{R}$, and $v_{\mathbf{R}}$ is the real part of v, thought of as a section of TM. Then $v_{\mathbf{R}} = \frac{1}{4}\mathrm{grad} u$, and $v_{\mathbf{R}}$ is a real holomorphic vector field, i.e. the one parameter group of diffeomorphisms $f_t : M \to M$ preserves the complex structure of M. We can then define the path of metrics

$$\omega_t = f_t^*(\omega),$$

and we can check that

$$\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t,$$

where

$$\dot{\varphi}_t = f_t^* u.$$

Then φ_t defines a geodesic line in $\mathcal{K}$ (see Exercise 4.4). The derivative of the Mabuchi functional along this line is given by

$$\begin{aligned}\frac{d}{dt}\mathcal{M}(\varphi_t) &= \int_M \dot{\varphi}_t(\hat{S} - S_t)\,\omega_t^n \\ &= \int_M f_t^* u\,(\hat{S} - f^*S(\omega))\,f^*(\omega^n) \\ &= \int_M u(\hat{S} - S(\omega))\,\omega^n \\ &= F(v),\end{aligned}$$

where $F(v)$ is the Futaki invariant of v. In other words the Mabuchi functional is linear along this geodesic line, with derivative given by the Futaki invariant.

Proposition 4.17. *The Mabuchi functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$ is convex along geodesics.*

Proof. Suppose that φ_t defines a geodesic and let us compute the second derivative of $\mathcal{M}(\varphi_t)$. By definition

$$\frac{d}{dt}\mathcal{M}(\varphi_t) = \int_M \dot{\varphi}_t(\hat{S} - S_t)\,\omega_t^n,$$

so

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{M}(\varphi_t) &= \int_M \big[\ddot{\varphi}_t(\hat{S} - S_t) + \dot{\varphi}_t(\mathcal{D}_t^*\mathcal{D}_t\dot{\varphi}_t - g_t^{j\bar{k}}\partial_j S_t \partial_{\bar{k}}\dot{\varphi}_t) \\ &\qquad + \dot{\varphi}_t(\hat{S} - S_t)\Delta_t\dot{\varphi}_t\big]\,\omega_t^n \\ &= \int_M \Big[|\mathcal{D}_t\dot{\varphi}_t|_t^2 + (\hat{S} - S_t)(\ddot{\varphi}_t - |\partial\dot{\varphi}_t|_t^2)\Big]\,\omega_t^n \\ &= \int_M |\mathcal{D}_t\dot{\varphi}_t|_t^2\,\omega_t^n \geqslant 0.\end{aligned}$$

Therefore $\mathcal{M}$ is convex along the path φ_t. $\square$

From this result a very appealing picture arises. We have a convex functional $\mathcal{M} : \mathcal{K} \to \mathbf{R}$, whose critical points are the cscK metrics in the class $[\omega]$. We can therefore at least heuristically expect a cscK metric to exist if and only if as we approach the "boundary" of $\mathcal{K}$, the derivative of $\mathcal{M}$ becomes positive. Since we are on an infinite dimensional space it is hard to make this picture rigorous, but we will see that the notion of K-stability can be seen as an attempt to encode this behavior "at infinity" of the functional $\mathcal{M}$.

Unfortunately it is difficult to construct geodesics in $\mathcal{K}$, and in fact it is possible to construct pairs of potentials in $\mathcal{K}$ on certain manifolds, which are not joined by a smooth geodesic (see Lempert-Vivas [27]). Nevertheless it is possible to show the existence of geodesics with enough regularity, that geometric conclusions can be drawn (see Chen-Tian [12]).

4.4 Extremal metrics on a ruled surface

In this section we will describe the construction of explicit extremal metrics on a ruled surface, due to Tønnesen-Friedman [53]. We will only do the calculation in a special case, but much more general results along these lines can be found in the work of Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman [1].

Let Σ be a genus 2 curve, and ω_Σ a Kähler metric on Σ with constant scalar curvature -2. By the Gauss-Bonnet theorem the area of Σ is 2π with this metric. Let L be a degree -1 holomorphic line bundle on Σ (i.e. $c_1(L)[\Sigma] = -1$), and let h be a metric on L with curvature form $F(h) = -\omega_\Sigma$.

We will construct metrics on the projectivization $X = \mathbf{P}(L \oplus \mathcal{O})$ over Σ, where $\mathcal{O}$ is the trivial line bundle. Thus X is a $\mathbf{CP}^1$-bundle over Σ. We will follow the method of Hwang-Singer [23]. First we construct metrics on the complement of the zero section in the total space of L, and then describe what is necessary to complete the metrics across the zero and infinity sections of X.

We will consider metrics of the form

$$\omega = p^*\omega_\Sigma + \sqrt{-1}\partial\overline{\partial} f(s), \tag{42}$$

where $p : L \to \Sigma$ is the projection map, $s = \log |z|^2_h$, and f is a strictly convex function which makes ω positive definite. Let us compute the metric ω in local coordinates. Choose a local holomorphic coordinate z on Σ and a fiber coordinate w for L, corresponding to a holomorphic trivialization around z. The fiberwise norm is then given by $|(z,w)|_h^2 = |w|^2 h(z)$ for some function h, and so our coordinate s is given by

$$s = \log|w|^2 + \log h(z).$$

Let us work at a point (z_0, w_0), in a trivialization such that $d\log h(z_0) = 0$. Then at this point

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial} f(s) &= f'(s)\sqrt{-1}\partial\bar{\partial}\log h + f''(s)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2} \\ &= f'(s)p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2},\end{aligned} \tag{43}$$

where we used that $-\sqrt{-1}\partial\bar{\partial}\log h$ is the curvature of L. It follows that

$$\omega = (1+f'(s))p^*\omega_\Sigma + f''(s)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}, \tag{44}$$

and so

$$\omega^2 = \frac{1}{|w|^2}(1+f'(s))f''(s)p^*\omega_\Sigma \wedge (\sqrt{-1}dw\wedge d\bar{w}).$$

We can check that if we now use a different trivialization for the line bundle in which $\tilde{w} = g(z)w$ for a holomorphic function z, then the same formula for ω^2 holds, so this formula holds at every point. It follows that the Ricci form of ω is

$$\begin{aligned}\rho &= -\sqrt{-1}\partial\bar{\partial}\log\left(\frac{1}{|w|^2}(1+f'(s))f''(s)\right) + p^*\rho_\Sigma \\ &= -\sqrt{-1}\partial\bar{\partial}\log\Big[(1+f'(s))f''(s)\Big] - 2p^*\omega_\Sigma,\end{aligned} \tag{45}$$

where $\rho_\Sigma = -2\omega_\Sigma$ is the Ricci form of Σ. We could at this point compute the scalar curvature of ω, but it is more convenient to change coordinates. From (44) we know that for ω to be positive, f must be strictly convex. We can therefore take the Legendre transform of f. The Legendre transform F is defined in terms of the variable $\tau = f'(s)$, by the formula

$$f(s) + F(\tau) = s\tau.$$

If $I \subset \mathbf{R}$ is the image of f', then F is a strictly convex function defined on I. The momentum profile of the metric is defined to be $\varphi : I \to \mathbf{R}$, where

$$\varphi(\tau) = \frac{1}{F''(\tau)}.$$

The following relations can be verified:

$$s = F'(\tau), \quad \frac{ds}{d\tau} = F''(\tau), \quad \varphi(\tau) = f''(s).$$

Using (44) and (45) we have

$$\begin{aligned} \omega &= (1+\tau)p^*\omega_\Sigma + \varphi(\tau)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} \\ \rho &= -\sqrt{-1}\partial\overline{\partial}\log\big[(1+\tau)\varphi(\tau)\big] - 2p^*\omega_\Sigma. \end{aligned} \tag{46}$$

A calculation now shows that the scalar curvature is given by

$$S(\tau) = -\frac{2}{1+\tau} - \frac{1}{1+\tau}\big[(1+\tau)\varphi\big]'', \tag{47}$$

where the primes mean derivatives with respect to τ.

We still need to understand when we can complete the metric across the zero and infinity sections. We will just focus on the metric in the fiber directions, which according to (44) is given by

$$f''(s)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}.$$

Let us write $r = |w|$, so $s = 2\log r$. The condition that this metric extends across $w = 0$ is that f'' has the form

$$f''(s) = c_2 r^2 + c_3 r^3 + c_4 r^4 + \ldots.$$

Then, since $d/ds = \frac{r}{2} d/dr$ we have

$$f'''(s) = c_2 r^2 + \frac{3}{2}c_3 r^3 + 2c_4 r^4 + \ldots,$$

and since $f''(s) = \varphi(\tau)$, and $f'''(s) = \varphi'(\tau)\varphi(\tau)$, we have

$$\varphi'(\tau) = 1 + O(r).$$

In particular if the range of τ is an interval (a, b), then

$$\lim_{\tau \to a} \varphi(\tau) = 0, \qquad \lim_{\tau \to a} \varphi'(\tau) = 1.$$

A similar computation can be done as $w \to \infty$, by changing coordinates to w^{-1}, showing that

$$\lim_{\tau \to b} \varphi(\tau) = 0, \qquad \lim_{\tau \to b} \varphi'(\tau) = -1.$$

Note also that by (46) the metric will be positive definite as long as $(1+\tau)$ and $\varphi(\tau)$ are positive on $[a,b]$. For simplicity we can take the interval $[0,m]$ for some $m>0$. The value of m determines the Kähler class of the resulting metric. Viewing X as a $\mathbf{CP}^1$-bundle over Σ, the space $H^2(X,\mathbf{R})$ is generated by Poincaré duals of a fiber C, and the infinity section S_∞, which is the image of the subbundle $L\oplus\{0\}\subset L\oplus\mathcal{O}$ under the projection map to the projectivization $X=\mathbf{P}(L\oplus\mathcal{O})$. We have the following intersection formulas:

$$C\cdot C=0,\quad S_\infty\cdot S_\infty=1,\quad C\cdot S_\infty=1.$$

The Kähler class of the metric can then be determined by computing the areas of C and S_∞. The area of C is given by

$$\int_{\mathbf{C}\setminus\{0\}} f''(s)\frac{\sqrt{-1}dw\wedge d\bar{w}}{|w|^2}=2\pi(\lim_{s\to\infty}f'(s)-\lim_{s\to-\infty}f'(s))=2\pi m,$$

while the area of the infinity section S_∞ is

$$(1+m)\int_\Sigma \omega_\Sigma=2\pi(1+m).$$

It follows that if we denote by $\mathcal{L}_m$ the Poincaré dual to the Kähler class of ω, then

$$\mathcal{L}_m=2\pi(C+mS_\infty).$$

The final thing to check is when the metric is extremal, i.e. when is $\mathrm{grad}^{1,0}S(\tau)$ holomorphic. We can compute that

$$\mathrm{grad}^{1,0}S(\tau)=S'(\tau)w\frac{\partial}{\partial w},$$

which is a holomorphic vector field if and only if $S'(\tau)$ is constant. So ω is extremal if and only if $S''(\tau)=0$.

The end result is the following theorem, which follows from the more general results in Hwang-Singer [23].

Theorem 4.18. *Suppose that $\varphi:[0,m]\to\mathbf{R}$ is a smooth function which is positive on $(0,m)$ and satisfies the boundary conditions*

$$\varphi(0)=\varphi(m)=0,\quad \varphi'(0)=1,\quad \varphi'(m)=-1. \tag{48}$$

Then by the above construction we obtain a metric on X, in the Kähler class Poincaré dual to $\mathcal{L}_m=2\pi(C+mS_\infty)$, whose scalar curvature is given by

$$S(\tau)=-\frac{2}{1+\tau}-\frac{1}{1+\tau}\big[(1+\tau)\varphi\big]''.$$

The metric is extremal if and only if $S''(\tau)=0$.

We can now construct extremal metrics, by solving the ODE $S''(\tau) = 0$ for $\varphi : [0, m] \to \mathbf{R}$ satisfying the boundary conditions (48). The equation to be solved is

$$\frac{-1}{1+\tau}\Big(2 + \big[(1+\tau)\varphi\big]''\Big) = A + B\tau,$$

for some A, B. This equation can easily be integrated, using the boundary conditions, and we obtain

$$\varphi(\tau) = \frac{\tau(m-\tau)}{m(m^2+6m+6)(1+\tau)}\big[\tau^2(2m+2) + \tau(-m^2+4m+6) \\ + m^2 + 6m + 6\big].$$

This will only give rise to a metric, if $\varphi(\tau) > 0$ for all $\tau \in (0, m)$. This happens only if $m < k_1$, where $k_1 \approx 18.889$ is the positive root of $m^4 - 16m^3 - 52m^2 - 48m - 12$. We have therefore constructed extremal metrics with non-constant scalar curvature on the $\mathbf{CP}^1$-bundle X, in the Kähler classes Poincaré dual to $\mathcal{L}_m$ for $m < k_1$.

It is interesting to see what happens as $m \to k_1$. At $m = k_1$ the solution $\varphi(\tau)$ aquires a zero in $(0, m)$ (Figure 1 shows the graph of φ when $m = 17$). Geometrically this corresponds to the fiber metrics degenerating in such a way that the diameter becomes unbounded, but the area remains bounded. In other words the fibers break up into two pieces, each with an end asymptotic to a hyperbolic cusp. We will see later that X does not admit an extremal metric when $m \geqslant k_1$.

4.5 Exercises

Exercise 4.1. Show that the space $\mathfrak{h}$ defined in Section 4.2 is closed under the Lie bracket.

Exercise 4.2. Give an example of a compact Kähler manifold M, and a holomorphic section v of $T^{1,0}M$ such that $v \notin \mathfrak{h}$.

Exercise 4.3. Suppose that we define the Mabuchi functional as follows. For any $\varphi \in \mathcal{K}$ let φ_t be a path in $\mathcal{K}$ such that $\varphi_0 = 0$ and $\varphi_1 = \varphi$, and define

$$\mathcal{M}(\varphi) = \int_0^1 \int_M \dot{\varphi}_t(\hat{S} - S_t)\omega_t^n \, dt,$$

where $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$ and S_t is the scalar curvature of ω_t. Check directly that this is well defined, i.e. the integral is independent of the path φ_t that we choose connecting 0 and φ.

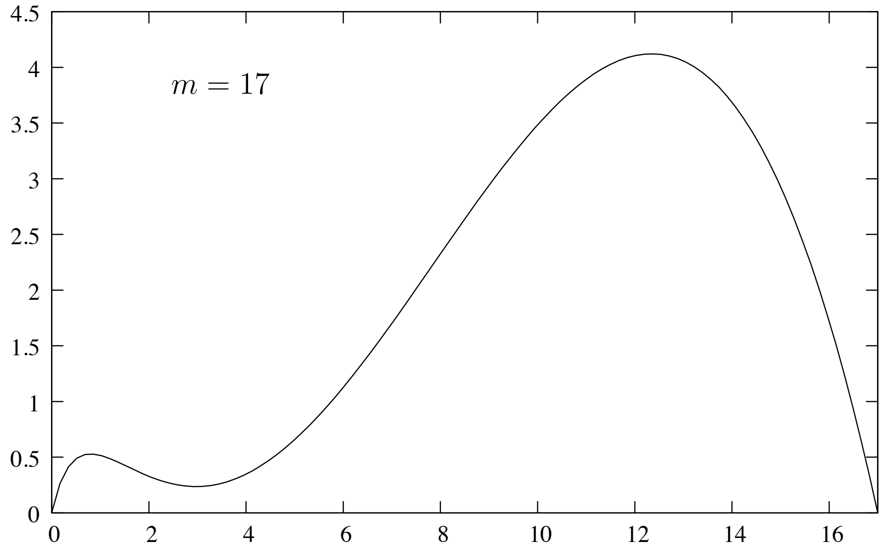

Figure 1: The momentum profile for the extremal metric when $m = 17$.

Exercise 4.4. Verify the claim in Example 4.16, that the family φ_t is a geodesic. As a first step check that in the notation of that example we can choose $\dot{\varphi}_t = f_t^* u$.

Exercise 4.5. Suppose that ω_1, ω_2 are two extremal metrics in the same Kähler class on M. Assuming that there is a geodesic path connecting ω_1 and ω_2, prove that there is a biholomorphism $f : M \to M$ such that $f^*\omega_2 = \omega_1$.

Exercise 4.6. Let ω be an extremal metric on a compact Kähler manifold M. Use the implicit function theorem to show that there exists an extremal metric in every Kähler class on M which is sufficiently close to $[\omega]$ (this is a theorem of LeBrun-Simanca [25]). At first you should assume that M has no holomorphic vector fields, which simplifies the problem substantially.

5 Moment maps and geometric invariant theory

In this section we will see that the search for extremal metrics in a Kähler class can be fit into a general framework, by interpreting the scalar curvature as a moment map for an infinite dimensional Hamiltonian action. The Calabi functional then becomes the norm squared of the moment map, and many of the calculations in the previous section can be reinterpreted from this point of view. For us the most important result about moment maps will be the link with geometric invariant theory provided by the Kempf-Ness theorem. For a much more thorough treatment of this subject see Mumford-Fogarty-Kirwan [36], or see Thomas [48] for an exposition with extremal metrics in mind.

5.1 Moment maps

Let (M, ω) be a compact Kähler manifold with Kähler metric g. We could work more generally with a symplectic manifold, but it is convenient to have the Kähler structure. The Hamiltonian construction assigns a vector field X_h on M to any smooth function $h : M \to \mathbf{R}$, satisfying

$$dh(Y) = -\omega(X_h, Y) = -\iota_{X_h}\omega(Y),$$

where ι_{X_h} is the contraction with X_h (contracting the first component). In terms of the metric g we have $dh(Y) = -g(JX_h, Y)$, so

$$X_h = J\mathrm{grad}\, h,$$

using the Riemannian gradient.

Lemma 5.1. *We have $L_{X_h}\omega = 0$, where L_{X_h} is the Lie derivative. In other words the one-parameter group of diffeomorphisms generated by X_h preserves the form ω.*

Proof. The Lie derivative satisfies the formula

$$L_{X_h}\omega = d(\iota_{X_h}\omega) + \iota_{X_h} d\omega.$$

This can be checked easily for 2-forms of the type $f\, dg \wedge dh$ and extended to arbitrary 2-forms by linearity. Since $d\omega = 0$ we have

$$L_{X_h}\omega = d(-dh) = 0,$$

using that $\iota_{X_h} = -dh$. □

Note that while X_h preserves ω, it does not preserve the metric g in general, unless X_h is a real holomorphic vector field (i.e. $L_{X_h}J = 0$ for the complex structure J).

Suppose now that a connected Lie group G acts on M, preserving the form ω. The derivative of the action gives rise to a Lie algebra map

$$\rho : \mathfrak{g} \to \mathrm{Vect}(M),$$

where $\mathfrak{g}$ is the Lie algebra of G and $\mathrm{Vect}(M)$ is the space of vector fields on M. Roughly speaking the action of G is called Hamiltonian, if each of the vector fields in the image of ρ arises from the Hamiltonian construction.

Definition 5.2. The action of G on M is Hamiltonian, if there exists a G-equivariant map

$$\mu : M \to \mathfrak{g}^*,$$

to the dual of the Lie algebra of $\mathfrak{g}$, such that for any $\xi \in \mathfrak{g}$ the function $\langle \mu, \xi \rangle$ is a Hamiltonian function for the vector field $\rho(\xi)$:

$$d\langle \mu, \xi \rangle = -\omega(\rho(\xi), \cdot).$$

The action of G on $\mathfrak{g}^*$ is by the coadjoint action. The map μ is called a moment map for the action.

Equivalently, the action is Hamiltonian if there is a G-equivariant lift $m : \mathfrak{g} \to C^\infty(M)$ of the map ρ, where G acts on $\mathfrak{g}$ by the adjoint action. In the diagram below, Ham refers to the Hamiltonian construction.

$$\begin{tikzcd} & C^\infty(M) \arrow[d, "\mathrm{Ham}"] \\ \mathfrak{g} \arrow[ur, dashed, "m"] \arrow[r, "\rho"'] & \mathrm{Vect}(M) \end{tikzcd}$$

For any given Hamiltonian vector field X the possible lifts to $C^\infty(M)$ all differ by addition of constants. The G-equivariance requires a consistent choice of such lifts. In practice this is usually easily achieved by choosing a "natural" normalization for the Hamiltonian functions, for example requiring them to have average zero if M is compact. On the other hand there are cases when the G-equivariance cannot be achieved.

The moment map is important in constructing quotients of symplectic manifolds. In the above set-up, with a Hamiltonian action of G on M and a choice of moment map μ, the symplectic quotient is defined to be $\mu^{-1}(0)/G$. If the action of G on $\mu^{-1}(0)$ is free, then this quotient inherits a natural symplectic structure from M. If M is Kähler, and the group G acts by isometries, then the quotient will inherit a Kähler structure. The basic idea is that at $x \in \mu^{-1}(0)$ the tangent space $T_x\mu^{-1}(0)$ is the kernel of $d\mu_x$, but from the definitions this is the orthogonal complement $(JT_xGx)^\perp$, where Gx is the G-orbit of x. We therefore have an identification

$$T_x\left(\mu^{-1}(0)/G\right) = (T_xGx \oplus JT_xGx)^\perp.$$

This is a complex subspace of T_xM, and the restrictions of the complex structure and the symplectic form define the Kähler structure on $\mu^{-1}(0)/G$. For more details on this see McDuff-Salamon [35].

Example 5.3. Consider the action $U(1) \curvearrowright \mathbf{C}$, by multiplication, and let $\omega = \sqrt{-1}dz \wedge d\bar{z} = 2dx \wedge dy$ be the standard Kähler form on $\mathbf{C}$. The action is generated by the vector field

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

so

$$\iota_X\omega = -2x - 2y.$$

So $h = x^2 + y^2$ satisfies $dh = -\iota_X\omega$. In other words,

$$\mu(z) = |z|^2$$

is a moment map for this action, after a suitable identification of $\mathfrak{u}(1)^*$ with $\mathbf{R}$. Other moment maps are given by $\mu(z) + c$ for any $c \in \mathbf{R}$.

Example 5.4. Generalizing the previous example, let $U(n) \curvearrowright \mathbf{C}^n$ be the standard action. For any given $A \in \mathfrak{u}(n)$ which generates a circle action, we can diagonalize A and apply the result of the previous example to each factor of $\mathbf{C}$. If A is diagonal with eigenvalues $\sqrt{-1}\lambda_i$ for $\lambda_i \in \mathbf{R}$, then a Hamiltonian function for the vector field X_A generated by A is given by

$$z = (z_1, \ldots, z_n) \mapsto \lambda_1|z_1|^2 + \ldots + \lambda_n|z_n|^2 = -\sqrt{-1}\bar{z}^tAz.$$

The same formula then holds for any A. This means that a moment map for the action is given by

$$\mu : \mathbf{C}^n \to \mathfrak{u}(n)^*$$
$$(z_1, \ldots, z_n) \mapsto \sqrt{-1}z_i\bar{z}_j,$$

where $\sqrt{-1}z_i\bar{z}_j$ defines a matrix in $\mathfrak{u}(n)$, and we identify $\mathfrak{u}(n)^* \simeq \mathfrak{u}(n)$ using the pairing $\langle A, B\rangle = -\mathrm{Tr}(AB)$.

Example 5.5. Consider now the action $U(n+1) \curvearrowright \mathbf{CP}^n$, which preserves the Fubini-Study form. $\mathbf{CP}^n$ is obtained from $\mathbf{C}^{n+1}$ as a symplectic quotient, with respect to the diagonal action of $U(1)$. More precisely, we choose the moment map

$$z \mapsto |z|^2 - 1$$

for this $U(1)$-action on $\mathbf{C}^{n+1}$. Then

$$\mathbf{CP}^n = (\{|z|^2 - 1 = 0\} \subset \mathbf{C}^{n+1})/U(1),$$

and the Fubini-Study form is the induced form on this quotient. The moment map on $\mathbf{CP}^n$ is therefore the moment map on $\mathbf{C}^{n+1}$ restricted to the subset where $|z|^2 = 1$. So we obtain the moment map

$$\begin{aligned} \mu : \mathbf{CP}^n &\to \mathfrak{u}(n+1)^* \\ [Z_0 : Z_1 : \ldots : Z_n] &\mapsto \frac{\sqrt{-1} Z_i \overline{Z}_j}{|Z|^2}. \end{aligned}$$

Here again $\mathfrak{u}(n+1)$ is identified with its dual using the pairing $-\mathrm{Tr}(AB)$. If we restrict the action to $SU(n+1)$, then the resulting moment map μ_{SU} is just the projection of μ onto $\mathfrak{su}(n+1)$, i.e.

$$\mu_{SU}([Z_0 : \ldots : Z_n]) = \frac{\sqrt{-1} Z_i \overline{Z}_j}{|Z|^2} - \frac{\sqrt{-1}}{n+1}\mathrm{Id},$$

where Id is the identity matrix. We can view $\mathbf{CP}^n$ as a coadjoint orbit in $\mathfrak{su}(n+1)^*$, and then μ_{SU} is the identity map.

Example 5.6. Consider the diagonal action $SU(2) \curvearrowright \mathrm{Sym}^n \mathbf{CP}^1$, on unordered n-tuples of points on $\mathbf{CP}^1$. We can identify $\mathfrak{su}(2)^*$ with $\mathbf{R}^3$, and $\mathbf{CP}^1$ with the unit sphere in $\mathbf{R}^3$ (as a coadjoint orbit). Under these identifications a moment map for the action is given by

$$\begin{aligned} &\mu : \mathrm{Sym}^n \mathbf{CP}^1 \to \mathbf{R}^3 \\ &\mu(x_1, \ldots, x_n) = x_1 + \ldots + x_n. \end{aligned}$$

This means that zeros of the moment map are given by n-tuples of points whose center of mass is the origin.

5.2 The scalar curvature as a moment map

In this section we will see that the scalar curvature can be viewed as a moment map. This was discovered by Fujiki [19] and Donaldson [14], and it not only sheds new light on the developments in Section 4 but it will motivate the definition of K-stability. We will only sketch the construction, but the details can be found in [14] and also Tian [52].

Now let (M,ω) be a symplectic manifold. This means that ω is a closed, non-degenerate 2-form. For simplicity we will assume that $H^1(M,\mathbf{R})=0$. Recall that an almost complex structure on M is an endomorphism $J : TM \to TM$ such that $J^2 = -Id$, where Id is the identity map. We say that an almost complex structure J is compatible with ω if the tensor

$$g_J(X,Y) = \omega(X,JY)$$

is symmetric and positive definite, i.e. it defines a Riemannian metric. If J is integrable, then (M,J) is a complex manifold and the metric g_J is Kähler. Define the infinite dimensional space

$$\mathscr{J} = \{\text{almost complex structures on } M \text{ compatible with } \omega\}.$$

The tangent space at a point J is given by

$$T_J\mathscr{J} = \left\{ \begin{array}{c} A : TM \to TM \ \text{ such that } AJ + JA = 0, \\ \text{and } \omega(X,AY) = \omega(Y,AX) \text{ for all } X,Y \end{array} \right\}.$$

If $A \in T_J\mathscr{J}$ then also $JA \in T_J\mathscr{J}$, and this defines a complex structure on $\mathscr{J}$. We can also define an inner product

$$\langle A,B\rangle_J = \int_M \langle A,B\rangle_{g_J} \frac{\omega^n}{n!},$$

for $A,B \in T_J\mathscr{J}$, which gives rise to a Hermitian metric on $\mathscr{J}$. Combining these structures we have a Kähler form on $\mathscr{J}$, given at the point J by

$$\Omega_J(A,B) = \langle JA,B\rangle_J.$$

We now let

$$\mathscr{G} = \{\text{group of Hamiltonian symplectomorphisms of } (M,\omega)\}.$$

These are the time-one flows of time dependent Hamiltonian vector fields. Using the Hamiltonian construction, the Lie algebra $\mathrm{Lie}(\mathscr{G})$ can be identified with the functions on M with zero integral, $C_0^\infty(M)$. The group $\mathscr{G}$ acts on $\mathscr{J}$ by pulling back complex structures, and this action preserves the Kähler form Ω.

Theorem 5.7 (Fujiki [19] and Donaldson [14])**.** *The action of $\mathscr{G}$ on $\mathscr{J}$ is Hamiltonian, and a moment map is given by*

$$\begin{aligned} \mu : \mathscr{J} &\to \mathrm{Lie}(\mathscr{G})^* \\ J &\mapsto S(J) - \hat{S}, \end{aligned}$$

where $S(J)$ is the scalar curvature of g_J when J is integrable, and $\hat{S}$ is the average of $S(J)$, which does not depend on J. The function $S(J) - \hat{S}$ is thought of as an element of the dual of $\mathrm{Lie}(\mathscr{G}) \cong C_0^\infty(M)$ *through the L^2 product on M.*

Note that if J is not integrable then $S(J)$ is the "Hermitian scalar curvature" of g_J, which is not the same as the Riemannian scalar curvature. In any case, the theorem means that finding constant scalar curvature Kähler metrics amounts to finding integrable J with $\mu(J) = 0$.

Let us unwind what it means for μ to be a moment map. For any $J \in \mathscr{J}$ there are two linear operators

$$\begin{aligned} P : C_0^\infty(M) &\to T_J\mathscr{J} \\ Q : T_J\mathscr{J} &\to C_0^\infty(M). \end{aligned}$$

The map P is given by the infinitesimal action of $\mathrm{Lie}(\mathscr{G})$ on $\mathcal{J}$. This can be written as

$$P(H) = L_{X_H} J,$$

where L_{X_H} is the Lie derivative with respect to the Hamiltonian vector field X_H. The map Q is the derivative of the map taking J to the Hermitian scalar curvature of g_J, so we can write

$$Q(A) = DS_J(A).$$

To say that μ is a moment map simply means that for all $A \in T_J\mathscr{J}$ and $H \in C_0^\infty(M)$ we have

$$\langle Q(A), H\rangle_J = -\Omega_J(P(H), A) = \langle JA, P(H)\rangle_{L^2},$$

where on the right we just have the L^2-product on functions. In other words the theorem means that $Q^* = -JP$ for the adjoint Q^* of Q.

Note that in previous lectures we were always fixing a complex structure since we were working on a fixed complex manifold, and we were varying the Kähler metric ω. Now we seem to be doing something rather different, fixing the form ω, and varying the complex structure instead. These two points of view can be related to each other as follows. For any symplectic form ω and compatible complex structure J let us write $g(J,\omega)$ for the corresponding Kähler metric. If J, J' are two complex structures and $J' = f^*J$ for a diffeomorphism J, then

$$g(J',\omega) = f^*g(J,(f^{-1})^*\omega). \tag{49}$$

Now if $f \in \mathscr{G}$, then this means that the two metrics are isometric. To obtain something non-trivial, we need to consider the complexification $\mathscr{G}^c$ of $\mathscr{G}$. While this complexified group does not exist, we can at least complexify the Lie algebra

$$\mathrm{Lie}(\mathscr{G}^c) = C_0^\infty(M, \mathbf{C}),$$

and since $\mathscr{J}$ has a complex structure, we can also naturally complexify the infinitesimal action. This complexified infinitesimal action gives rise to a foliation on $\mathscr{J}$, whose leaves can be thought of as the orbits of $\mathscr{G}^c$.

Claim 5.8. *If $J \in \mathscr{J}$ is integrable, then the $\mathscr{G}^c$-orbit of J (or rather just the orbit of the imaginary part of $\mathscr{G}^c$) can be identified with the set of Kähler metrics on (M,J) in the class $[\omega]$.*

To see this at an infinitesimal level, let $J \in \mathscr{J}$ be integrable, suppose that $H \in C_0^\infty(M)$, and let us see what the action of $\sqrt{-1}H$ looks like on $T_J\mathscr{J}$. When J is integrable, then

$$JP(H) = JL_{X_H}J = L_{JX_H}J,$$

so infinitesimally the action of $\sqrt{-1}H$ means flowing along the vector field JX_H. By the observation (49) we obtain the same metric this way as if we fix J instead, and flow ω along the vector field $-JX_H$. The variation of ω is then

$$-L_{JX_H}\omega = -d\iota_{JX_H}\omega = 2\sqrt{-1}\partial\overline{\partial}H, \tag{50}$$

and so flowing along this vector field amounts to moving ω in its Kähler class.

The upshot of all this is that at least formally, the problem of finding a cscK metric in the Kähler class $[\omega]$ on the complex manifold (M,J) is equivalent to finding a zero of the moment map μ, for the action of $\mathscr{G}$, in the orbit J under the complexified action. This is a general question that can be posed for other Hamiltonian actions on Kähler manifolds, and in the finite dimensional case the Kempf-Ness theorem gives an answer. In the next few sections we will develop the background needed for this theorem.

5.3 Geometric invariant theory

Suppose that $M \subset \mathbf{CP}^n$ is a projective variety (see below for definitions), and a complex Lie group $G \subset SL(n+1, \mathbf{C})$ acts on M by biholomorphisms. Geometric invariant theory gives a way of constructing a quotient M/G which is also a projective variety. The basic idea is that M/G should be characterized by the requirement, that

$$\text{“functions on } M/G\text{”} = \text{“}G\text{-invariant functions on } M\text{”}.$$

Example 5.9. Before giving more precise definitions, let us look at a simple example, which illustrates some of the ideas, although it does not fit precisely in the framework that we are considering since here we are working with affine varieties instead of projective ones. Suppose that $\mathbf{C}^*$ acts on $\mathbf{C}^2$ with the action

$$\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y).$$

There are 3 types of orbits:

(i) $xy = t$ for $t \neq 0$. These are closed 1-dimensional orbits.

(ii) $x = y = 0$. This is a closed 0-dimensional orbit.

(iii) $x = 0, y \neq 0$, or $x \neq 0, y = 0$. These are two 1-dimensional orbits, whose closures contain the origin.

The orbit space is not Hausdorff, because the closure of the orbits of type (iii) contain the orbit (ii). However if we discard the non-closed orbits (iii), then the remaining orbits are parametrized by $\mathbf{C}$. In terms of functions, the $\mathbf{C}^*$-invariant functions on $\mathbf{C}^2$ are

$$\mathbf{C}[x, y]^{\mathbf{C}^*} = \mathbf{C}[xy] \cong \mathbf{C}[t],$$

and so the space of functions of the quotient should be $\mathbf{C}[t]$. Therefore the quotient $\mathbf{C}^2/\mathbf{C}^*$ from this point of view should also be $\mathbf{C}$.

Looking ahead a little, this example also illustrates the relationship with the moment map very clearly. The action of the maximal compact subgroup $U(1) \subset \mathbf{C}^*$ on $\mathbf{C}^2$ is Hamiltonian with respect to the standard symplectic form, and a moment map is given by (see Example 5.4)

$$\mu(x, y) = |x|^2 - |y|^2.$$

The symplectic quotient $\mu^{-1}(0)/U(1)$ also equals $\mathbf{C}$, and $\mu^{-1}(0)$ intersects each closed orbit of $\mathbf{C}^*$ in a $U(1)$-orbit.

In order to define the GIT quotient in general, we need a quick review of some basic ideas in algebraic geometry.

Definition 5.10. (a) A projective variety $X \subset \mathbf{CP}^n$ is the zero set of a collection of homogeneous polynomials $f_1, \ldots, f_k$, which is irreducible, i.e. it cannot be written as a non-trivial union of two such zero sets.

(b) The homogeneous coordinate ring of X is the graded ring (graded by degree)

$$R(X) = \mathbf{C}[x_0, \ldots, x_n]/I,$$

where I is the ideal generated by the homogeneous polynomials vanishing on X. Since X is irreducible, the ideal I is prime (i.e. if $fg \in I$, then $f \in I$ or $g \in I$). Equivalently the ring $R(X)$ has no zero-divisors.

Conversely any homogeneous prime ideal $I \subset \mathbf{C}[x_0, \ldots, x_n]$ gives rise to a projective variety, as long as $I \neq (x_0, \ldots, x_n)$ (in which case the zero set would be empty). The Nullstellensatz in commutative algebra implies that there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{homogeneous prime ideals} \\ \text{in } \mathbf{C}[x_0, \ldots, x_n] \\ \text{except } (x_0, \ldots, x_n) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{projective subvarieties} \\ \text{of } \mathbf{CP}^n \end{array} \right\}.$$

It is often convenient to work on the level of functions, and forget the way that our variety is embedded in projective space. The correspondence in this case is

$$\left\{ \begin{array}{c} \text{finitely generated graded } \mathbf{C}\text{-algebras,} \\ \text{generated in degree one,} \\ \text{without zero-divisors} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{projective} \\ \text{varieties} \end{array} \right\}, \tag{51}$$

although to make this correspondence one-to-one, we would have to define equivalence relations on both sets. In this correspondence a projective variety is mapped to its homogeneous coordinate ring. For the converse direction, if R is a finitely generated graded $\mathbf{C}$-algebra, generated in degree one, then there is a surjective grading preserving map

$$\mathbf{C}[x_0, \ldots, x_n] \to R,$$

mapping each x_i to a degree 1 generator. If I is the kernel of this map, then,

$$\mathbf{C}[x_0, \ldots, x_n]/I \cong R,$$

and I is prime since R has no zero-divisors. The vanishing set of the homogeneous elements in I is the projective variety $X \subset \mathbf{CP}^n$, corresponding to R. Let us call this projective variety $\mathrm{Proj}(R)$.

Remark 5.11. In the theory of schemes, the above correspondences are extended by allowing arbitrary homogeneous ideals as opposed to just prime ideals, and correspondingly, arbitrary finitely generated $\mathbf{C}$-algebras, not just those without zero-divisors (even more generally one does not need to work over a field, but we do not need this). We will see later how these schemes arise naturally, and how we can think of them geometrically.

With this background we can proceed to define the GIT quotient.

Definition 5.12. A complex Lie group G is reductive, if it is the complexification of a maximal compact subgroup $K \subset G$. For example $SL(n, \mathbf{C})$ is reductive with maximal compact subgroup $SU(n)$. Similarly the group $\mathbf{C}^*$ is reductive with maximal compact subgroup $U(1)$. On the other hand the additive group $\mathbf{C}$ is not reductive since it has no non-trivial compact subgroups at all.

Suppose that a complex reductive group G acts on a projective variety $X \subset \mathbf{CP}^n$, and the action is induced by a representation

$$G \to SL(n+1, \mathbf{C}).$$

Through the dual action on $\mathbf{C}[x_0, \ldots, x_n]$, this induces an action of G on the homogeneous coordinate ring $R(X)$. Using that G is a reductive group, one can show that the ring of invariants $R(X)^G$ is finitely generated. Let us write

$$R(X)^G = \bigoplus_{k \geqslant 0} R(X)^G_k,$$

where $R(X)^G_k$ is the degree k piece. To get a projective variety, we would like to replace $R(X)^G$ with a $\mathbf{C}$-algebra which is generated in degree 1. For this, one shows that there is a $d > 0$ such that the subalgebra

$$\widetilde{R}(X)^G = \bigoplus_{k \geqslant 0} R(X)^G_{kd}$$

is generated by elements in $R(X)^G_d$. Changing the grading so that $R(X)^G_{kd}$ is the degree k piece in $\widetilde{R}(X)^G$, we obtain a $\mathbf{C}$-algebra generated in degree one, and we define the GIT quotient to be

$$X /\!\!/ G = \mathrm{Proj}\widetilde{R}(X)^G.$$

Since $\widetilde{R}(X)^G$ is a subalgebra of $R(X)$, it has no zero-divisors, and so $X /\!\!/ G$ is a projective variety.

While this definition is very simple, at least once the correspondence (51) has been established, it is unclear at this point what the quotient $X /\!\!/ G$ represents geometrically. To understand this, let us choose degree one generators $f_0, \ldots, f_k$ of $\widetilde{R}(X)^G$, and look at the map

$$\begin{aligned} q : X &\dashrightarrow \mathbf{CP}^k \\ p &\mapsto [f_0(p), \ldots, f_k(p)], \end{aligned}$$

which is only defined at points $p \in X$ at which there is at least one non-vanishing G-invariant function in R. Then the image of q is $X /\!\!/ G$, and q is the quotient map. The main points are therefore the following:

(i) The quotient $X /\!\!/ G$ parametrizes orbits on which there is at least one non-vanishing G-invariant function in R.

(ii) The quotient map $q : X \dashrightarrow X /\!\!/ G$ identifies any two orbits which cannot be distinguished by G-invariant functions in R.

This motivates the following definitions.

Definition 5.13. The set of semistable points $X^{ss} \subset X$ is defined by

$$X^{ss} = \left\{ p \in X \;\middle|\; \begin{array}{c} \text{there exists a non-constant homogeneous } f \in R(X)^G \\ \text{such that } f(p) \neq 0 \end{array} \right\}.$$

The set of stable points $X^s \subset X^{ss}$ is defined by

$$X^s = \left\{ p \in X^{ss} \;\middle|\; \begin{array}{c} \text{the stabiliser of } p \text{ in } G \text{ is finite,} \\ \text{and the orbit } G \cdot p \text{ is closed in } X^{ss} \end{array} \right\}.$$

Both X^s, X^{ss} are open subsets of X. The GIT quotient $X /\!\!/ G$ can be thought of as the quotient of X^{ss} by the equivalence relation that $p \sim q$ if $\overline{G \cdot p} \cap \overline{G \cdot q}$ is non-empty in X^{ss}. The role of the stable points X^s is that G has closed orbits on X^s, so a "geometric quotient" X^s/G exists, and this sits inside the GIT quotient $X /\!\!/ G$.

5.4 The Hilbert-Mumford criterion

Let us suppose as in the previous sections, that a complex reductive group $G \subset SL(n+1, \mathbf{C})$ acts on a projective variety $X \subset \mathbf{CP}^n$, where the action is induced by the natural action on $\mathbf{CP}^n$. In this section we will discuss a criterion for determining whether a given point $p \in X$ is stable or semistable.

For any $p \in X$, let us write $\hat{p} \in \mathbf{C}^{n+1} \setminus \{0\}$ for a lift of p with respect to the projection map $\mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$. We will write $G \cdot \hat{x}$ for the G orbit of $\hat{x}$ in $\mathbf{C}^{n+1}$.

Proposition 5.14. *(a) A point $p \in X$ is semistable if and only if $0 \notin \overline{G \cdot \hat{p}}$.*

(b) A point $p \in X$ is stable if and only if the orbit $G \cdot \hat{p}$ is closed in $\mathbf{C}^{n+1}$, and the stabilizer of p in G is finite.

Sketch of proof. (a) If $p \in X$ is semistable, then there is a homogeneous G-invariant polynomial f of positive degree, which does not vanish at $\hat{p}$. The G-invariance implies that f is a non-zero constant on the orbit closure $\overline{G \cdot \hat{p}}$, so the origin cannot be in this closure.

Conversely if $0 \notin \overline{G \cdot \hat{p}}$, then one can show that there exists a G-invariant homogeneous polynomial f distinguishing the disjoint, closed, G-invariant sets 0 and $\overline{G \cdot \hat{p}}$. This polynomial f does not vanish at $\hat{p}$, so p is semistable.

(b) Suppose first that $p \in X$ is stable. If $G\cdot\hat{p}$ is not closed, then the closure contains another orbit $G \cdot \hat{q}$, for some $q \in \overline{G \cdot p}$. Then necessarily $q \in X$, and q is semistable. This contradicts that the orbit of p in X^{ss} is closed.

Conversely suppose that $G\cdot p$ is not closed in X^{ss}, and let $q \in \overline{G \cdot p} \cap X^{ss}$ such that $q \notin G \cdot p$. Then there is a non-constant homogeneous G-invariant polynomial f which does not vanish at p and q, and we can assume that $f = 1$ on $G \cdot \hat{p}$ and $G \cdot \hat{q}$. From this one shows that the closure of $G \cdot \hat{p}$ contains $G \cdot \hat{q}$, and in particular $G \cdot \hat{p}$ is not closed. □

Since we will need it later, we define a third notion of stability at this point.

Definition 5.15. A point $p \in X$ is polystable, if the orbit $G \cdot \hat{p}$ is closed. Note that stable $\Rightarrow$ polystable $\Rightarrow$ semistable but the converses are false in general. We will see later that the closure of every semistable orbit contains a unique polystable orbit. In other words the GIT quotient $X /\!\!/ G$ can be thought of as parametrizing the polystable orbits.

The Hilbert-Mumford criterion essentially says that in order to check whether an orbit $G \cdot \hat{p}$ is closed, it is enough to check this for all one-parameter subgroups $\mathbf{C}^* \subset G$. In practice this is very useful, since the action of a one-parameter group can always be diagonalized, and this makes it possible to do some explicit calculations. In addition the Hilbert-Mumford criterion will motivate the definition of K-stability.

Definition 5.16. Suppose that $\lambda : \mathbf{C}^* \hookrightarrow G$ is a one-parameter subgroup. For any $p \in X$, define the weight $\mu(x, \lambda)$ as follows. First, let $q \in X$ be the limit

$$q = \lim_{t\to 0} \lambda(t) \cdot p$$

(we will see below that this limit exists). The point q is necessarily fixed by the one-parameter subgroup λ, so there exists an integer w, such that $\lambda(t) \cdot \hat{q} = t^w \hat{q}$ for all t. We define $\mu(p, \lambda) = -w$.

A useful way to think of this is the following. Given a one-parameter subgroup of G, acting on $\mathbf{C}^{n+1}$, we can write $\mathbf{C}^{n+1}$ as a sum of weight spaces

$$\mathbf{C}^{n+1} = \bigoplus_{i=1}^{k} V(w_i),$$

where each w_i is an integer, $\lambda(t) \cdot v = t^{w_i} v$ for $v \in V(w_i)$, and $k \leqslant n+1$. We can arrange that $w_1 < w_2 < \ldots < w_k$. Given $p \in X$, we can write $\hat{p} = \hat{p}_1 + \ldots + \hat{p}_k$, where $\hat{p}_i \in V(w_i)$. If l is the smallest index for which $\hat{p}_l$ is non-zero, then the limit $q = \lim_{t\to 0} \lambda(t) \cdot p$ is obtained by letting $\hat{q} = \hat{p}_l$. Then $\mu(p, \lambda) = -w_l$.

Theorem 5.17 (Hilbert-Mumford criterion)**.**

(a) $p \in X$ *is semistable* $\Leftrightarrow \mu(p, \lambda) \geqslant 0$ *for all 1-parameter subgroups* λ.

(b) $p \in X$ *is polystable* $\Leftrightarrow \mu(p, \lambda) > 0$ *for all 1-parameter subgroups* λ *for which* $\lim_{t\to 0} \lambda(t) \cdot p \notin G \cdot p$.

(c) $p \in X$ *is stable* $\Leftrightarrow \mu(p, \lambda) > 0$ *for all 1-parameter subgroups* λ.

Remarks on the proof. One direction of the result is fairly straight forward. For instance for part (a) suppose that λ is a one-parameter subgroup such that $\mu(p, \lambda) < 0$. Following the discussion before the theorem, we can write $\hat{p} = \hat{p}_1 + \ldots + \hat{p}_k$ in terms of the weight spaces of λ. Then

$$\lambda(t) \cdot \hat{p} = t^{w_1}\hat{p}_1 + \ldots + t^{w_k}\hat{p}_k,$$

and $\mu(p, \lambda) < 0$ means that the smallest weight w_i for which $\hat{p}_i \neq 0$ is positive. This means that λ acts on $\hat{p}$ with only positive weights, and so

$$\lim_{t \to 0} \lambda(t) \cdot \hat{p} = 0.$$

Therefore $0 \in \overline{G \cdot \hat{p}}$, and so p cannot be semistable.

The difficult part of the theorem is to show the converse. One method is to reduce the problem to the case of a torus action, for which the statement can be checked directly. □

Example 5.18. This example is the algebro-geometric counterpart to Example 5.6, where we looked at the action of $SU(2)$ on n-tuples of points on $\mathbf{CP}^1$. Let

$$V_n = \{\text{homogeneous degree } n \text{ polynomials in } x, y\} \cong \mathbf{C}^{n+1},$$

and let $X = \mathbf{P}(V_n)$. By identifying a polynomial with its zero set on $\mathbf{CP}^1$, we can think of X as the space of unordered n-tuples of points on $\mathbf{CP}^1$.

The group $SL(2, \mathbf{C})$ acts on V_n by

$$(M \cdot P)(x, y) = P(M^{-1}(x, y)),$$

where $M \in SL(2, \mathbf{C}), P \in V_n$, and $M^{-1}(x, y)$ is the standard action of $SL(2, \mathbf{C})$ on $\mathbf{C}^2$. In terms of n-tuples of points, this action corresponds to moving the points around on $\mathbf{CP}^1$, using the usual action of $SL(2, \mathbf{C})$ on $\mathbf{CP}^1$.

Let us determine the stable points for this action. Let λ be a one-parameter subgroup of $SL(2, \mathbf{C})$. We can choose a basis u, v for $\mathbf{C}^2$, such that in this basis λ is given by

$$\lambda(t) = \begin{pmatrix} t^w & 0 \\ 0 & t^{-w} \end{pmatrix},$$

for some integer $w > 0$. The induced action on a polynomial $P(u, v) = a_0 u^n + a_1 u^{n-1} v + \ldots + a_n v^n$ is

$$(\lambda(t) \cdot P)(u, v) = t^{-nw} a_0 u^n + t^{-(n-2)w} a_1 u^{n-1} v + \ldots + t^{nw} a_n v^n.$$

Writing $[P] \in \mathbf{P}(V_n)$ for the point in projective space corresponding to P, we have

$$\begin{aligned} [\lambda(t) \cdot P] &= [t^{-nw} a_0 u^n + \ldots t^{nw} a_n v^n] \\ &= [a_k u^{n-k} v^k + t^{2w} a_{k+1} u^{n-k-1} v^{k+1} + \ldots + t^{(2n-2k)w} a_n v^n], \end{aligned}$$

where k is the smallest index for which $a_k \neq 0$. Therefore

$$\lim_{t \to 0} [\lambda(t) \cdot P] = [a_k u^{n-k} v^k].$$

Since

$$\lambda(t) \cdot (a_k u^{n-k} v^k) = t^{2k-n} a_k u^{n-k} v^k,$$

we have $\mu([P], \lambda) = n - 2k$. By the Hilbert-Mumford criterion we need $k < n/2$ for $[P]$ to be stable. Since k was the smallest index for which $a_k \neq 0$, this means that P is not divisible by $v^{n/2}$, i.e. in terms of the n-tuple of zeros of P, the point $[1:0]$ has multiplicity less than $n/2$. Choosing different 1-parameter subgroups amounts to looking at different points, so we obtain that an n-tuple of points is stable, if and only if no point is repeated $n/2$ times.

In a similar way one can determine that an n-tuple is semistable if and only if no point is repeated more than $n/2$ times. Finally an n-tuple is polystable if either it is stable, or it consists of just 2 points with multiplicity $n/2$. If n is odd, then all three notions of stability coincide.

Recall that in Example 5.6 we saw that if we look at the action of $SU(2)$ on such n-tuples on $\mathbf{CP}^1$, then zeros of the moment map are those n-tuples of points, whose center of mass is the origin (thinking of $\mathbf{CP}^1$ as the unit sphere $S^2 \subset \mathbf{R}^3$). The Kempf-Ness theorem which we will discuss later implies that an n-tuple is polystable, if and only if its $SL(2, \mathbf{C})$-orbit contains a zero of the moment map. In other words we can move an n-tuple of points on $\mathbf{CP}^1$ into a balanced position (with center of mass the origin) by an element in $SL(2, \mathbf{C})$ if and only if no $n/2$ points conincide, or the n-tuple consists of just 2 points with multiplicity $n/2$. One direction of this is clear: if too many points coincide, then we certainly cannot make the center of mass be the origin.

5.5 The Hilbert polynomial and flat limits

In this section we collect some background material in algebraic geometry which we will need later on. Recall that for a projective variety $X \subset \mathbf{CP}^n$ we defined the homogeneous coordinate ring

$$R(X) = \mathbf{C}[x_0, \ldots, x_n]/I,$$

where I is the ideal generated by the homogeneous polynomials which vanish on X. This is a graded ring, whose degree d piece $R_d(X)$ is image of the degree d polynomials under the quotient map. Each $R_d(X)$ is a finite dimensional vector space, and the Hilbert function of X is defined by

$$H_X(d) = \dim R_d(X).$$

A fundamental result is that there is a polynomial $P_X(d)$, called the Hilbert polynomial, such that $H_X(d) = P_X(d)$ for sufficiently large d. The degree of P_X is the dimension of X. The polynomial P_X should be thought of as an invariant of X, and one of its crucial properties is that it does not change if we vary X in a nice enough family (the technical condition is that the family is "flat").

The Hilbert function can be defined more generally for any homogeneous ideal $I \subset \mathbf{C}[x_0, \ldots, x_n]$, by letting

$$H_I(d) = \dim \left(\mathbf{C}[x_0, \ldots, x_n]/I\right)_d,$$

where again we are taking the image of the degree d polynomials under the quotient map. Once again one can show that for large enough d, the Hilbert function $H_I(d)$ coincides with a polynomial $P_I(d)$.

We will now give a very special example of a flat family, which will be enough for our needs. Suppose that $I \subset \mathbf{C}[x_0, \ldots, x_n]$ is a homogeneous ideal, and that we have a one-parameter subgroup $\lambda : \mathbf{C}^* \hookrightarrow SL(n+1, \mathbf{C})$. For any polynomial f, we can define $\lambda(t) \cdot f$ by

$$(\lambda(t) \cdot f)(x_0, \ldots, x_n) = f\big(\lambda(t^{-1}) \cdot (x_0, \ldots, x_n)\big),$$

and it is easy to check that

$$I_t = \{\lambda(t) \cdot f \,|\, f \in I\}$$

is also a homogeneous ideal in $\mathbf{C}[x_0, \ldots, x_n]$. Geometrically the vanishing set of I_t is obtained by applying $\lambda(t)$ to the vanishing set of I.

Definition 5.19. The flat limit $I_0 = \lim_{t \to 0} I_t$ is defined as follows. We can decompose any $f \in I$ as $f = f_1 + \ldots + f_k$ into elements in distinct weight spaces for the $\mathbf{C}^*$-action λ on $\mathbf{C}[x_0, \ldots, x_n]$. Let us write $in(f)$ for the element f_i with the smallest weight, which we can think of as the "initial term" of f. Then I_0 is the ideal generated by the set of initial terms $\{in(f) \,|\, f \in I\}$.

For any ideal I let us write $(I)_d$ for the degree d piece of I. Then one can check that for our flat limit, the degree d piece $(I_0)_d$ of I_0 is the vector space spanned by $\{in(f) \mid f \in (I)_d\}$. From this it is not hard to see that $\dim(I_0)_d = \dim(I)_d$ for each d, so the Hilbert polynomials of I and I_0 are the same. In fact in this case even the Hilbert function is preserved in the limit, but that is not true for more general flat limits.

Example 5.20. A simple example is letting $I \subset \mathbf{C}[x,y,z]$ be the ideal $I = (xz - y^2)$, i.e. the ideal generated by the polynomial $xz - y^2$. The corresponding projective variety is a conic in $\mathbf{CP}^2$. Let us take the $\mathbf{C}^*$-action given by $\lambda(t) \cdot (x,y,z) = (tx, t^{-1}y, z)$. The dual action on functions gives

$$\lambda(t) \cdot (xz - y^2) = t^{-1}xz - t^2y^2.$$

The initial term is $in(xz - y^2) = xz$, and so the flat limit is

$$\lim_{t \to 0} \lambda(t) \cdot I = (xz).$$

The variety corresponding to (xz) is two lines intersecting in a point. In other words when taking the limit, the conic breaks up into two intersecting lines. While this limit is not irreducible, it is still the union of two projective varieties.

Note that in general by just taking the initial terms of a set of generators of the ideal, we might get a smaller ideal than the flat limit. In the example we are looking at here, we can check that the Hilbert polynomial of (xz) equals the Hilbert polynomial of $(xz - y^2)$, so (xz) has to be the flat limit. More generally one can use Gröbner bases to do these calculations.

Example 5.21. For a similar example let us take $I = (xz - y^2)$ again, but let $\lambda(t) \cdot (x,y,z) = (t^{-1}x, ty, z)$. Then

$$\lambda(t) \cdot (xz - y^2) = txz - t^{-2}y^2,$$

so now the initial term is $-y^2$, and the flat limit is

$$\lim_{t \to 0} \lambda(t) \cdot I = (y^2).$$

The zero set of (y^2) is a line in $\mathbf{CP}^2$, but it should be thought of as having multiplicity 2, or as being "thickened". The quotient ring $\mathbf{C}[x,y,z]/(y^2)$ has nilpotents, and the corresponding geometric object is a projective scheme.

The flat limits that we are considering arise when we try to form a GIT quotient of the space of all projective subvarieties in $\mathbf{CP}^n$. More precisely one can show that given a polynomial P, there is a projective scheme

$Hilb_{P,n}$, called the Hilbert scheme, parametrizing all projective subschemes of $\mathbf{CP}^n$ with Hilbert polynomial P. The idea is that if a homogeneous ideal $I \subset \mathbf{C}[x_0, \ldots, x_n]$ has Hilbert polynomial P, then there is a large number d (depending on P, n), such that the scheme corresponding to I is determined by the degree d piece of I, which we still denote by $(I)_d$ (technically for this one needs to restrict attention to "saturated" ideals - saturating an ideal does not change the corresponding scheme). This $(I)_d$ is simply a linear subspace of the degree d polynomials $\mathbf{C}[x_0, \ldots, x_n]_d$, such that

$$\dim \mathbf{C}[x_0, \ldots, x_n]_d/(I)_d = P(d).$$

Since the degree d piece of I determines the scheme, we obtain a map from the set of schemes with Hilbert polynomial P, to a certain Grassmannian of subspaces of a finite dimensional vector space. Roughly speaking the image of this map is the Hilbert scheme (although it has more structure than just being a subset).

The "moduli space" of projective varieties (or schemes) in $\mathbf{CP}^n$ with a given Hilbert polynomial should then be the GIT quotient

$$Hilb_{P,n} /\!\!/ SL(n+1, \mathbf{C}),$$

since acting by $SL(n+1, \mathbf{C})$ simply changes the embedding of a variety, not the variety itself. If we try to use the Hilbert-Mumford criterion to determine whether a given variety is stable (or semistable), then we naturally arrive at the notion of a flat limit under a $\mathbf{C}^*$-action which we defined above.

5.6 The Kempf-Ness theorem

Suppose now that $M \subset \mathbf{CP}^n$ is a projective submanifold, with a complex group $G \subset SL(n+1, \mathbf{C})$ acting on M. Let $K = G \cap SU(n+1)$, and assume that $K \subset G$ is a maximal compact subgroup. This means, on the level of Lie algebras, that

$$\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}.$$

Recall that

$$\begin{aligned} \mu_U : \mathbf{CP}^n &\to \mathfrak{u}(n+1)^* \\ [Z_0 : \ldots : Z_n] &\mapsto \frac{\sqrt{-1} Z_i \overline{Z}_j}{|Z|^2}, \end{aligned}$$

is a moment map for the $U(n+1)$-action on $\mathbf{CP}^n$. The restriction of this to M, projected to $\mathfrak{k}^*$, is a moment map

$$\mu : M \to \mathfrak{k}^*$$

for the action of K on M, with respect to the symplectic form given by the restriction of the Fubini-Study metric to M. Recall also that for $p \in M$ we let $\hat{p} \in \mathbf{C}^{n+1} \setminus \{0\}$ be a lift, and say that p is polystable for the action of G, if the G-orbit $G \cdot \hat{p} \subset \mathbf{C}^{n+1}$ is closed.

Theorem 5.22 (Kempf-Ness)**.** *A point $p \in M$ is polystable for the action of G, if and only if the orbit $G \cdot p$ contains a zero of the moment map μ. Moreover if p is polystable, then $G \cdot p \cap \mu^{-1}(0)$ is a single K-orbit.*

Sketch of proof. The main point of the proof is introducing the following function

$$\begin{aligned} \mathcal{M} : G/K &\to \mathbf{R} \\ [g] &\mapsto \log |g \cdot \hat{p}|^2, \end{aligned}$$

where $|\cdot|$ is the Euclidean norm on $\mathbf{C}^{n+1}$, and $[g]$ denotes the coset gK. Note that since $K \subset SU(n+1)$, the K action preserves the norm, so $\mathcal{M}$ is well-defined.

The space G/K can be endowed with a Riemannian metric, so that it is a non-positively curved symmetric space. The geodesics are given by one-parameter subgroups $[e^{t\sqrt{-1}\xi} g]$ for $\xi \in \mathfrak{k}$ and $g \in G$. The two main points are

(i) $[g]$ is a critical point of $\mathcal{M}$ if and only if $\mu(g \cdot x) = 0$.

(ii) $\mathcal{M}$ is convex along geodesics in G/K.

The orbit $G \cdot \hat{p}$ is closed precisely when the norm $|g \cdot \hat{p}|$ goes to infinity as g goes to infinity, and this corresponds to the function $\mathcal{M}$ being proper. Because of the convexity, this happens exactly when $\mathcal{M}$ has a critical point.

To see (i), we need to compute the derivative of $\mathcal{M}$. Fix a $g \in G$, and write $g \cdot \hat{p} = Z$, and choose a skew-Hermitian matrix $A \in \mathfrak{k}$. We have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathcal{M}(e^{t\sqrt{-1}A} g) &= \left.\frac{d}{dt}\right|_{t=0} \log |e^{t\sqrt{-1}A} g \cdot \hat{p}|^2 \\ &= \frac{-\sqrt{-1} Z^T \overline{AZ} + \sqrt{-1}(AZ)^T \overline{Z}}{|Z|^2} \\ &= \frac{2\sqrt{-1}\,\overline{Z}^T AZ}{|Z|^2} \\ &= -2\langle \mu(g \cdot p), A \rangle, \end{aligned} \tag{52}$$

where we used that A is skew-Hermitian and we are using the pairing $\langle A, B \rangle = -\mathrm{Tr}(AB)$. It follows that $[g]$ is a critical point of $\mathcal{M}$ if and only if $\mu(g \cdot p) = 0$.

To see (ii) we need to compute the second derivative:

$$\begin{aligned}\frac{d^2}{dt^2}\bigg|_{t=0} \mathcal{M}(e^{t\sqrt{-1}A}g) &= -2\,\frac{d}{dt}\bigg|_{t=0} \langle \mu(e^{t\sqrt{-1}A}g\cdot p), A\rangle \\ &= 2g_{FS}(A_{g\cdot p}, A_{g\cdot p}) \geqslant 0,\end{aligned} \tag{53}$$

where g_{FS} is the Fubini-Study metric restricted to M, and by $A_{g\cdot p}$ we mean the tangent vector at $g\cdot p$ induced by the infinitesimal action of A. $\square$

We will shortly see that $\mathcal{M}$ is the finite dimensional analog of the Mabuchi functional from Section 4.3. There is also an analog of the Futaki invariant, given as follows. For $p \in M$, let $G_p \subset G$ be the stabilizer of p, and $\mathfrak{g}_p$ its Lie algebra. The group G_p acts on the line spanned by $\hat{p}$, and we denote the infinitesimal action by the Lie algebra homomorphism

$$F : \mathfrak{g}_p \to \mathbf{C}.$$

We can compute F in terms of the moment map. Writing $Z = \hat{p}$ again, if $A \in \mathfrak{g}_p$, then $AZ = F(A)Z$ by definition. Then

$$\langle \mu(p), A\rangle = \frac{-\sqrt{-1}\overline{Z}^T AZ}{|Z|^2} = -\sqrt{-1}F(A),$$

so

$$F(A) = \sqrt{-1}\langle \mu(p), A\rangle. \tag{54}$$

In Section 5.2 we have seen that the scalar curvature arises as a moment map for the action of an infinite dimensional group $\mathscr{G}$, and zeros of this moment map correspond to constant scalar curvature Kähler metrics. Moreover, although the complexification $\mathscr{G}^c$ is not a group, one can make sense of what its orbits should be, and they correspond to Kähler metrics in a fixed Kähler class. Therefore, the problem of finding cscK metrics in a Kähler class can be formulated as finding zeros of the moment map in a $\mathscr{G}^c$-orbit. This is exactly the setup in the Kempf-Ness theorem, and we can see what the functions $\mathcal{M}$ and F above correspond to in the infinite dimensional setup. Without going into details, the symmetric space G/K corresponds to the space of Kähler metrics with Mabuchi's L^2 metric (the Killing form on $\mathfrak{k}$ is replaced by the L^2 product on functions), the function $\mathcal{M}$ corresponds to the Mabuchi functional (compare the formula (52) to the variation (41) of the Mabuchi functional), and F corresponds to the Futaki invariant (compare (54) to the definition (40) of the Futaki invariant).

5.7 Test-configurations and K-stability

In this section we will introduce the notion of K-stability. This was originally defined by Tian [51], and conjectured to characterize the existence of a Kähler-Einstein metric on a manifold with positive first Chern class. A more refined, algebro-geometric definition was introduced by Donaldson [16], which he conjectured to characterize the existence of a cscK metric. It is this definition which we will use.

The definition of K-stability is inspired by the Hilbert-Mumford criterion for stability in GIT, which we discussed in Section 5.4. Throughout this section we will work with a projective manifold $M \subset \mathbf{CP}^N$ with $\dim M = n$, although it is more natural to think of the pair (M, L), where $L = \mathcal{O}(1)|_M$. In general such a pair (M, L) of a projective manifold together with an ample line bundle L is called a polarized manifold.

Definition 5.23. A test-configuration for (M, L), of exponent $r > 0$, consists of an embedding $M \hookrightarrow \mathbf{CP}^{N_r}$ using a basis of sections of L^r, and a $\mathbf{C}^*$ subgroup of $GL(N_r + 1, \mathbf{C})$.

Given a test-configuration with $\lambda : \mathbf{C}^* \hookrightarrow GL(N_r + 1, \mathbf{C})$ being the $\mathbf{C}^*$-subgroup, we obtain a family of submanifolds $M_t \subset \mathbf{CP}^{N_r}$, with $M_t = \lambda(t) \cdot M$. This family can be extended across $t = 0$, by taking the flat limit

$$M_0 = \lim_{t \to 0} M_t,$$

according to Definition 5.19. Usually the definition of a test-configuration is formulated in terms of the resulting flat $\mathbf{C}^*$-equivariant family over $\mathbf{C}$. By construction the flat limit M_0 is preserved by the $\mathbf{C}^*$-action λ, and by analogy with the Hilbert-Mumford criterion, we need to define a weight for this action. This weight is given by the Donaldson-Futaki invariant.

To define the weight, suppose that $X \subset \mathbf{CP}^N$ is a subscheme, invariant under a $\mathbf{C}^*$-action λ. Algebraically this means that we have a homogeneous ideal

$$I \subset \mathbf{C}[x_0, \ldots, x_N],$$

which is preserved by the dual action of λ. It follows that there is an induced $\mathbf{C}^*$-action on the homogeneous coordinate ring

$$R = \mathbf{C}[x_0, \ldots, x_N]/I,$$

and each degree k piece R_k is invariant. Let us write $d_k = \dim R_k$ for the Hilbert function of X, and let w_k be the total weight of the action on R_k.

As we discussed in section 5.5, for large k, d_k equals a polynomial, so we have

$$d_k = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$

for some constants a_0, a_1. The degree n is the dimensions of X. Similarly w_k equals a polynomial of degree $n+1$ for large k, so we can define constants b_0 and b_1 by

$$w_k = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}).$$

Definition 5.24. The Donaldson-Futaki invariant of the $\mathbf{C}^*$-action λ on X is defined to be

$$DF(X, \lambda) = \frac{a_1}{a_0} b_0 - b_1.$$

Note that in the literature sometimes the formula has the opposite signs, because of varying conventions on whether one takes the dual action or not. We will see in Section 6.3 that if X is smooth, and the $\mathbf{C}^*$-action is induced by a holomorphic vector field, then the Donaldson-Futaki invariant coincides with the differential geometric Futaki invariant introduced in Section 4.2.

Before defining K-stability we need to define when we consider a test-configuration to be trivial. Originally a test-configuration was defined to be trivial if the central fiber M_0 is biholomorphic to M. It was pointed out by Li-Xu [28] that this is not enough in general, unless one restricts attention to those test-configurations, whose total space is "normal". This is a condition on the type of singularities that can occur. Instead we give an alternative definition, relying on the norm of a test-configuration. Using the same notation as above, suppose that we have a $\mathbf{C}^*$-action λ on a subscheme $X \subset \mathbf{CP}^N$, and let us write A_k for the infinitesimal generator of the $\mathbf{C}^*$-action on the degree k-piece R_k of the homogeneous coordinate ring. Then $\mathrm{Tr}(A_k) = w_k$ in the notation above. Similarly to d_k and w_k, the function $\mathrm{Tr}(A_k^2)$ is a polynomial of degree $n+2$, and we define c_0 by

$$\mathrm{Tr}(A_k^2) = c_0 k^{n+2} + O(k^{n+1}).$$

The norm $\|\lambda\|$ of the $\mathbf{C}^*$-action λ is defined to be

$$\|\lambda\|^2 = c_0 - \frac{b_0^2}{a_0},$$

where a_0, b_0 are as above. In other words, $\|\lambda\|^2$ is the leading term in

$$\mathrm{Tr}\left(A_k - \frac{\mathrm{Tr}(A_k)}{d_k} Id\right)^2 = \|\lambda\|^2 k^{n+2} + O(k^{n+1}).$$

We can now give the definition of K-stability.

Definition 5.25. Let (M, L) be a polarized manifold. Given a test-configuration χ for (M, L), let us also write χ for the induced $\mathbf{C}^*$-action on the central fiber, so we have the norm $\|\chi\|$ and the Donaldson-Futaki invariant $F(\chi) = DF(M_0, \chi)$.

The pair (M, L) is K-semistable, if for every test-configuration χ we have $F(\chi) \geqslant 0$. If in addition $F(\chi) > 0$ whenever $\|\chi\| > 0$, then (M, L) is K-stable.

One version of the central conjecture in the field is the following.

Conjecture 5.26 (Yau-Tian-Donaldson)**.** *Let (M, L) be a polarized manifold, and suppose that M has discrete holomorphic automorphism group. Then M admits a cscK metric in $c_1(L)$ if and only if (M, L) is K-stable.*

There is also a version of the conjecture applicable when M has holomorphic vector fields, where cscK metrics are replaced by extremal metrics, see [47]. An example due to Apostolov-Calderbank-Gauduchon-Tønnesen-Friedman [1] suggests that the notion of K-stability needs to be strengthened, so as written the conjecture should be thought of more as a guiding principle.

One of our goals in this course will be to explain the proof of one direction of the conjecture, due to Stoppa [45], which built on the work of Donaldson [17] and Arezzo-Pacard [2] (see also Mabuchi [31]). In the Kähler-Einstein case the result is due to Tian [51] and Paul-Tian [38].

Theorem 5.27. *If M admits a cscK metric in $c_1(L)$ and has discrete automorphism group, then (M, L) is K-stable.*

Example 5.28. Let $(M, L) = (\mathbf{CP}^1, \mathcal{O}(1))$, and embed $M \hookrightarrow \mathbf{CP}^2$ using the sections Z_0^2, Z_0Z_1, Z_1^2 of $\mathcal{O}(2)$ as a conic $xz - y^2 = 0$. Consider the $\mathbf{C}^*$-action $\lambda(t) \cdot (x, y, z) = (tx, t^{-1}y, z)$ as in Example 5.20. The central fiber is given by $xz = 0$, and the dual action on functions has weights $(-1, 1, 0)$ on (x, y, z). In order to compute the Donaldson-Futaki invariant, let us write

$$S = \mathbf{C}[x, y, z],$$

$I = (xz)$, and $R = S/I$. We have an exact sequence

$$0 \longrightarrow S \xrightarrow{\cdot xz} S \longrightarrow R \longrightarrow 0,$$

where the second map is multiplication by the generator xz of the ideal I. Let us write S_k and R_k for the degree k pieces of S and R. Let $D_k = \dim S_k$, and let W_k be the total weight of the $\mathbf{C}^*$-action on S_k. Similarly write $d_k = \dim R_k$, and w_k for the total weight of the action on R_k. From the exact sequence, for $k \geqslant 2$ we have

$$d_k = D_k - D_{k-2},$$

and since the weight of the action on xz is -1, we have

$$w_k = W_k - (W_{k-2} - D_{k-2}).$$

By symmetry of the weights on x, y we must have $W_k = 0$ for all k, and in addition

$$D_k = \binom{k+2}{2} = \frac{1}{2}(k^2 + 3k + 2).$$

It follows that

$$\begin{aligned} d_k &= 2k + 1 \\ w_k &= \frac{1}{2}k^2 - \frac{1}{2}k. \end{aligned}$$

By the definition of the Donaldson-Futaki invariant, we have

$$F(\lambda) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} = \frac{3}{4}.$$

In particular $F(\lambda) > 0$, so this test-configuration does not destabilize $\mathbf{CP}^1$. This is consistent since of course the Fubini-Study metric gives a cscK metric in $c_1(\mathcal{O}(1))$. More general calculations can be done similarly, except when the ideal I is not generated by a single polynomial, then instead of the short exact sequence that we used, one would need to use a longer "free resolution" of the homogeneous coordinate ring.

5.8 Exercises

Exercise 5.1. Let M_n be the set of $n \times n$ complex matrices, equipped with the Euclidean metric under the identification $M_n = \mathbf{C}^{n^2}$. The unitary matrices $U(n)$ act on M_n by conjugation, preserving this metric. I.e. $A \in U(n)$ acts by $M \mapsto A^{-1}MA$. Find a moment map

$$\mu : M_n \to \mathfrak{u}(n)^*$$

for this action (normalize it so that $\mu(0)$ is the zero matrix).

Exercise 5.2. With the notation of the previous question, note that $GL(n, \mathbf{C})$ also acts on M_n by conjugation (it does not preserve the metric though).

(a) Identify the closed orbits of this action.

(b) By the Kempf-Ness theorem, every closed $GL(n, \mathbf{C})$-orbit contains a $U(n)$-orbit of zeroes of the moment map μ. What linear algebra statement does this correspond to?

Exercise 5.3. Let $X \subset \mathbf{CP}^2$ be the conic $xz - y^2 = 0$. Consider the $\mathbf{C}^*$-action $\lambda(t) \cdot (x, y, z) = (t^{-1}x, ty, z)$ (this is the opposite of what we did in class). Find the Donaldson-Futaki invariant of the corresponding test-configuration.

6 The Bergman kernel

In this section we discuss the asymptotic expansion of the Bergman kernel. This provides a crucial link between algebraic and differential geometry, and it is the basis of many results in the field. Our goal in this section will be to use it to prove Donaldson's Theorem 6.11, providing a lower bound on the Calabi functional in terms of the Futaki invariants of test-configurations [17]. A corollary of this result is that if a X admits a cscK metric in $c_1(L)$ for a line bundle L, then (X, L) is K-semistable.

6.1 The Bergman kernel

Let M be a compact complex manifold, and L a positive line bundle over M. Suppose that L is equipped with a Hermitian metric h, which has positive curvature form $F(h)$. Let us define the Kähler form $\omega = \frac{1}{2\pi}F(h)$, so that $\omega \in c_1(L)$.

The Hermitian metric h induces a natural Hermitian metric on the space of holomorphic sections $H^0(M, L)$. For $s, t \in H^0(M, L)$ we define

$$\langle s, t\rangle_{L^2} = \int_M \langle s, t\rangle_h \frac{\omega^n}{n!}.$$

Definition 6.1. Choose an orthonormal basis $\{s_0, \ldots, s_N\}$ of $H^0(M, L)$. The Bergman kernel of the Hermitian metric h is the function

$$\begin{aligned} B_h : M &\to \mathbf{R} \\ x &\mapsto \sum_{i=0}^{N} |s_i(x)|_h^2. \end{aligned}$$

It is easy to check that B_h is independent of the orthonormal basis chosen. Note that B_h is actually the restriction to the diagonal of the full Bergman kernel, but we will not need this. Also, the function B_h is often called the density of states.

An alternative definition is given by the following.

Lemma 6.2. *For any $x \in M$ we have*

$$B_h(x) = \sup\{|s(x)|_h^2 \ : \ \|s\|_{L^2} = 1\}.$$

Proof. It is clear that $B_h(x) \geqslant |s(x)|_h^2$ for any s such that $\|s\|_{L^2} = 1$, by considering any orthonormal basis containing s.

For the converse inequality, write $E_x \subset H^0(M, L)$ for the space of sections vanishing at x. If $B_h(x) > 0$, then there must be a section which does not vanish at x, and so E_x has codimension 1. Let s be in the orthogonal complement of E_x, such that $\|s\|_{L^2} = 1$. Then it follows from the definition that $B_h(x) = |s(x)|_h^2$ since every section orthogonal to s vanishes at x. $\square$

The Bergman kernel has the following geometric interpretation.

Lemma 6.3. *Suppose that the map*

$$\begin{aligned}\varphi : M &\to \mathbf{CP}^N \\ x &\mapsto [s_0(x) : \ldots : s_N(x)]\end{aligned}$$

is defined on all of M, where $\{s_i\}$ is an orthonormal basis of $H^0(M, L)$. Then

$$\varphi^*\omega_{FS} = 2\pi\omega + \sqrt{-1}\partial\overline{\partial}\log B_h,$$

where ω_{FS} is the Fubini-Study metric.

Proof. On the subset of M where $s_0 \neq 0$, we have

$$\begin{aligned}\varphi^*\omega_{FS} &= \sqrt{-1}\partial\overline{\partial}\log\left(1 + \left|\frac{s_1}{s_0}\right|^2 + \ldots + \left|\frac{s_N}{s_0}\right|^2\right) \\ &= \sqrt{-1}\partial\overline{\partial}\log\left(1 + \frac{|s_1|_h^2}{|s_0|_h^2} + \ldots + \frac{|s_N|_h^2}{|s_0|_h^2}\right) \\ &= -\sqrt{-1}\partial\overline{\partial}\log|s_0|_h^2 + \sqrt{-1}\partial\overline{\partial}\log B_h \\ &= 2\pi\omega + \sqrt{-1}\partial\overline{\partial}\log B_h,\end{aligned}$$

since $2\pi\omega$ is the curvature of h. The same argument works on the open sets where $s_i \neq 0$ for each i, and these cover M. $\square$

The Hermitian metric h on L induces a metric h^k on L^k, and we get a corresponding Kähler form $k\omega$. Repeating the above construction with this metric, we obtain a function B_{h^k} on M. The key result is the asymptotic behavior of this function as $k \to \infty$.

Theorem 6.4. *As $k \to \infty$, we have*

$$B_{h^k} = 1 + \frac{S(\omega)}{4\pi}k^{-1} + O(k^{-2}), \tag{55}$$

where $S(\omega)$ is the scalar curvature of ω. More precisely, there are functions $a_0, a_1, \ldots$ on M such that $a_0 = 1$ and $a_1 = \frac{1}{4\pi} S(\omega)$, and for any integers $p, q \geqslant 0$ there is a constant C, such that

$$\left\| B_{h^k} - \sum_{i=0}^{p} a_i k^{-i} \right\|_{C^q} \leqslant C k^{-p-1}.$$

This theorem is due to Tian [49], Ruan [40], Zelditch [57], Lu [30], Catlin [9], and by now there is a large literature on it. In the next section we will only prove a simpler statement, giving the pointwise asymptotics (55). For now we look at some simple applications.

The original motivation of Tian was the following result, which implies that any Kähler metric in $c_1(L)$ can be approximated by "algebraic" metrics, obtained as pull-backs of Fubini-Study metrics under projective embeddings. The result follows immediately from Lemma 6.3 and the previous Theorem.

Corollary 6.5. *For large k, an orthonormal basis of $H^0(M, L^k)$ gives a map $\varphi_k : M \to \mathbf{CP}^{N_k}$, where $N_k + 1 = \dim H^0(M, L^k)$, and*

$$\frac{1}{k} \varphi_k^* \omega_{FS} - 2\pi\omega = O(k^{-2}), \quad \textit{in } C^\infty.$$

Another application is the following special case of the Hirzebruch-Riemann-Roch theorem.

Corollary 6.6. *As $k \to \infty$, we have*

$$\dim H^0(M, L^k) = k^n \int_M \frac{\omega^n}{n!} + \frac{k^{n-1}}{4\pi} \int_M S(\omega) \frac{\omega^n}{n!} + O(k^{n-2}).$$

Proof. We integrate the expansion (55) over M, remembering that $\{s_i\}$ form an orthonormal basis for $H^0(M, L^k)$. This means that

$$\begin{aligned} \dim H^0(M, L^k) &= \int_M \sum_{i=0}^{N_k} |s(x)|_h^2 \frac{(k\omega)^n}{n!} \\ &= \int_M B_{h^k} \frac{(k\omega)^n}{n!} \\ &= \int_M \left(1 + \frac{S(\omega)}{4\pi} k^{-1} + O(k^{-2})\right) \frac{k^n \omega^n}{n!} \\ &= k^n \int_M \frac{\omega^n}{n!} + \frac{k^{n-1}}{4\pi} \int_M S(\omega) \frac{\omega^n}{n!} + O(k^{n-2}). \end{aligned}$$

□

6.2 Proof of the asymptotic expansion

In this section we will prove the pointwise expansion (55), using the "peaked section" method of Tian [49], following the exposition of Donaldson [18].

Fix a point $x \in M$. The basic idea is to try to construct a holomorphic section σ of L^k, such that $\|\sigma\|_{L^2} = 1$, and which is L^2-orthogonal to all holomorphic sections of L^k which vanish at x. If we could do this, then we would have $B_{h^k}(x) = |\sigma(x)|_h^2$. Although we cannot do this exactly, for large k it is possible to construct sections σ which are almost orthogonal to the sections which vanish at x. This is enough to calculate $B_{h^k}(x)$ up to an $O(k^{-2})$ error.

We can choose holomorphic coordinates w^i centered at x, such that $2\pi\omega = \sqrt{-1}\partial\overline{\partial}\varphi$, where

$$\varphi(w) = |w|^2 - \frac{1}{4}R_{i\bar{j}k\bar{l}}w^i\bar{w}^j w^k \bar{w}^l + Q(w) + P(w).$$

Here $R_{i\bar{j}k\bar{l}}$ is the curvature tensor of $2\pi\omega$ at x, Q is a quintic polynomial, and $P(w) = O(|w|^6)$. Suppose the w^i are defined in a small neighborhood $B \subset M$ of x. For simplicity we can assume that $B = \{|w| < 1\}$. We can choose a holomorphic section s of L over B, such that

$$|s|_h^2 = e^{-\varphi},$$

and for each k, we will use s^k to trivialize the bundle L^k over B.

Introduce coordinates $z^i = \sqrt{k}w^i$, and let $\Phi(z) = k\varphi(w)$. Then

$$\Phi(z) = |z|^2 - \frac{k^{-1}}{4}R_{i\bar{j}k\bar{l}}z^i\bar{z}^j z^k \bar{z}^l + k^{-3/2}Q(z) + kP(k^{-1/2}z), \tag{56}$$

and Φ is a Kähler potential for $2\pi k\omega$ in B. In terms of z we have $B = \{|z| < \sqrt{k}\}$. For large k, we can choose a cutoff function χ such that $\chi(z) = 1$ for $|z| < k^{1/5}$ and $\chi(z) = 0$ for $|z| > 2k^{1/5}$, and moreover $|\nabla\chi| < 1$. The reason for choosing $k^{1/5}$ is that on the ball $\{|z| < 2k^{1/5}\}$, we can make $\Phi(z)$ be arbitrarily close to $|z|^2$ by choosing k to be large enough. In particular for large k the metric $\sqrt{-1}\partial\overline{\partial}\Phi$ will be very close to the Euclidean metric.

The truncated function χs^k can be extended by zero outside B, and so it can be thought of as a global section σ_0 of L^k. It is not holomorphic, but

$$\overline{\partial}\sigma_0 = \overline{\partial}(\chi s^k) = (\overline{\partial}\chi)s^k$$

is supported in the annulus $\{k^{1/5} \leqslant |z| \leqslant k^{2/5}\}$, and

$$|\overline{\partial}\sigma_0|_{h^k}^2 \leqslant |s^k|_{h^k}^2$$

on this annulus. It follows that for large k

$$\|\overline{\partial}\sigma_0\|_{L^2}^2 \leqslant C\int_{k^{1/5}<|z|<2k^{1/5}} e^{-\frac{1}{2}|z|^2}\,dV = \varepsilon(k), \tag{57}$$

where by $\varepsilon(k)$ we mean a function of k that decays faster than any power of k. By construction $|\sigma_0(x)|_{h^k}^2 = 1$.

The next task is to show that σ_0 can be perturbed to obtain a global holomorphic section of L^k. This uses the so called Hörmander technique. The main point is the following.

Lemma 6.7. *Let $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ be the $\overline{\partial}$-Laplacian on L^k-valued $(0,1)$-forms, where on L^k we use the metric h^k, and on forms we use $2\pi k\omega$. If k is sufficiently large, then for any L^k-valued $(0,1)$-form α we have*

$$\langle\Delta_{\overline{\partial}}\alpha,\alpha\rangle_{L^2} \geqslant \frac{1}{2}\|\alpha\|_{L^2}^2. \tag{58}$$

Proof. This essentially follows from the Weitzenbock formula

$$\Delta_{\overline{\partial}} = \overline{\nabla}^*\overline{\nabla} + \mathrm{Ric} + F,$$

where Ric and F are endomorphisms obtained from the Ricci curvature of $k\omega$, and the curvature form of h^k respectivaly. The point is that F is the identity, whereas as $k\to\infty$, Ric goes to zero. The details are as follows.

Let us write g for the metric $2\pi k\omega$, which is also the curvature of h^k. First note that

$$\langle\Delta_{\overline{\partial}}\alpha,\alpha\rangle_{L^2} = \|\overline{\partial}\alpha\|_{L^2}^2 + \|\overline{\partial}^*\alpha\|_{L^2}^2.$$

In local coordinates let us write $\alpha = \alpha_{\bar{i}}d\bar{z}^i$, where the $\alpha_{\bar{i}}$ are sections of L^k. Let us work at a point x in normal coordinates for g. Then

$$\overline{\partial}\alpha = \sum_{j,k}\nabla_{\bar{j}}\alpha_{\bar{k}}d\bar{z}^j\wedge d\bar{z}^k = \sum_{j<k}(\nabla_{\bar{j}}\alpha_{\bar{k}} - \nabla_{\bar{k}}\alpha_{\bar{j}})d\bar{z}^j\wedge d\bar{z}^k,$$

where ∇ is the Chern connection on L^k coupled with the Levi-Civita connection on $(0,1)$-forms. Since the $d\bar{z}^j\wedge d\bar{z}^k$ form an orthonormal basis, we have

$$\begin{aligned}
|\overline{\partial}\alpha|^2 &= \sum_{j<k}|\nabla_{\bar{j}}\alpha_{\bar{k}}|^2 + |\nabla_{\bar{k}}\alpha_{\bar{j}}|^2 - \nabla_{\bar{j}}\alpha_{\bar{k}}\overline{\nabla_{\bar{k}}\alpha_{\bar{j}}} - \nabla_{\bar{k}}\alpha_{\bar{j}}\overline{\nabla_{\bar{j}}\alpha_{\bar{k}}} \\
&= \sum_{j,k}|\nabla_{\bar{j}}\alpha_{\bar{k}}|^2 - \nabla_{\bar{j}}\alpha_{\bar{k}}\overline{\nabla_{\bar{k}}\alpha_{\bar{j}}} \\
&= g^{p\bar{j}}g^{q\bar{k}}\nabla_{\bar{j}}\alpha_{\bar{k}}\overline{(\nabla_{\bar{p}}\alpha_{\bar{q}} - \nabla_{\bar{q}}\alpha_{\bar{p}})},
\end{aligned}$$

where we have used summation convention in the last line, and the metric h is implied in the pairing of L with $\overline{L}$. The last expression is coordinate invariant, so this is $|\overline{\partial}\alpha|^2$ even if we are not in normal coordinates. Note also that

$$\overline{\partial}^*\alpha = -g^{j\bar{k}}\nabla_j\alpha_{\bar{k}}.$$

We therefore have

$$\begin{aligned}
\|\overline{\partial}\alpha\|^2_{L^2} + \|\overline{\partial}^*\alpha\|^2_{L^2} &= \int_M g^{p\bar{j}}g^{q\bar{k}}\big[\nabla_{\bar{j}}\alpha_{\bar{k}}\overline{(\nabla_{\bar{p}}\alpha_{\bar{q}} - \nabla_{\bar{q}}\alpha_{\bar{p}})} + \nabla_q\alpha_{\bar{k}}\overline{\nabla_j\alpha_{\bar{p}}}\big]\,dV \\
&= \|\overline{\nabla}\alpha\|^2_{L^2} + \int_M g^{p\bar{j}}g^{q\bar{k}}\big[(\nabla_q\nabla_{\bar{j}}\alpha_{\bar{k}})\overline{\alpha_{\bar{p}}} - (\nabla_{\bar{j}}\nabla_q\alpha_{\bar{k}})\overline{\alpha_{\bar{p}}}\big]\,dV \\
&\geqslant \int_M g^{p\bar{j}}g^{q\bar{k}}\big[R^{\bar{m}}_{\bar{k}q\bar{j}}\alpha_{\bar{m}}\overline{\alpha_{\bar{p}}} + F_{q\bar{j}}\alpha_{\bar{k}}\overline{\alpha_{\bar{p}}}\big]\,dV \\
&= \int_M \big[g^{p\bar{j}}g^{q\bar{k}}R_{p\bar{k}}\alpha_{\bar{j}}\overline{\alpha_{\bar{q}}} + g^{p\bar{k}}\alpha_{\bar{k}}\overline{\alpha_{\bar{p}}}\big]\,dV.
\end{aligned}$$

Since the Ricci form $R_{p\bar{k}}$ is invariant under scaling the metric ω, we have that $R_{p\bar{k}} = O(k^{-1}g)$, since g is the metric $2\pi k\omega$. For sufficiently large k, we will then have

$$\|\overline{\partial}\alpha\|^2_{L^2} + \|\overline{\partial}^*\alpha\|^2_{L^2} \geqslant \frac{1}{2}\|\alpha\|^2_{L^2},$$

which is what we wanted to prove. □

It follows from this result, that for large k the operator $\Delta_{\overline{\partial}}$ has trivial kernel, and since it is self-adjoint, it is invertible (by a result analogous to Theorem 2.12). Let us now return to our section σ_0. Define

$$\sigma = \sigma_0 - \overline{\partial}^*\Delta^{-1}_{\overline{\partial}}\overline{\partial}\sigma_0.$$

Since $\Delta_{\overline{\partial}}$ commutes with $\overline{\partial}$ and $\overline{\partial}^*$, it is easy to see that

$$\overline{\partial}\sigma = 0,$$

so σ is a holomorphic section of L^k. A priori it could be the zero section, however the estimates (57) and (58) imply that for large k

$$\begin{aligned}
\|\sigma - \sigma_0\|^2_{L^2} &= \|\overline{\partial}^*\Delta^{-1}_{\overline{\partial}}\overline{\partial}\sigma_0\|^2_{L^2} \\
&= \langle\Delta^{-1}_{\overline{\partial}}\overline{\partial}\sigma_0, \overline{\partial}\sigma_0\rangle \\
&\leqslant 2\|\overline{\partial}\sigma_0\|^2_{L^2} \\
&= \varepsilon(k).
\end{aligned} \tag{59}$$

At the same time σ_0 is holomorphic on the ball $|z| < k^{1/5}$, so $\sigma - \sigma_0$ is also holomorphic on this ball. The L^2 bound, and the estimate from Corollary 2.3 for harmonic functions implies that $|\sigma - \sigma_0|^2_{h^k}(x) = \varepsilon(k)$. This implies that

$$|\sigma(x)|^2_{h^k} = 1 + \varepsilon(k),$$

so if k is large enough, σ does not vanish at x.

Next we want to show that σ is approximately orthogonal to every holomorphic section which vanishes at x.

Lemma 6.8. *There is a constant C independent of k, such that*

$$|\langle \tau, \sigma \rangle_{L^2}| \leqslant Ck^{-1}\|\tau\|_{L^2}$$

for every holomorphic section $\tau \in H^0(M, L^k)$ vanishing at x.

Proof. Using the trivializing section s^k, we can think of τ as a holomorphic function of z, which vanishes at $z = 0$. Then by the mean value theorem we have

$$\int_{|z|<k^{1/5}} \tau(z) e^{-|z|^2}\, dV = 0, \tag{60}$$

where dV is the Euclidean volume form. We need to see that this differs from $\langle \tau, \sigma \rangle_{L^2}$ by at most $Ck^{-1}\|\tau\|_{L^2}$. First of all, by (59), we have

$$\langle \tau, \sigma \rangle_{L^2} = \langle \tau, \sigma_0 \rangle_{L^2} + \varepsilon(k)\|\tau\|_{L^2}.$$

Also, recall that $\sigma_0 = \chi s^k$, where χ was supported in the annulus $k^{1/5} < |z| < 2k^{1/5}$, where we can assume that $\Phi(z) > \frac{1}{2}|z|^2$, and $2\pi k\omega$ is uniformly equivalent to the Euclidean metric. It follows that

$$\langle \tau, \sigma \rangle_{L^2} = \varepsilon(k)\|\tau\|_{L^2} + \int_{|z|<k^{1/5}} \tau(z) e^{-\Phi(z)} \frac{1}{(2\pi)^n n!} (\sqrt{-1}\partial\bar{\partial}\Phi)^n$$

Using the expansion (56) combined with (60) for the leading term, we find that

$$\begin{aligned} |\langle \tau, \sigma \rangle_{L^2}| &\leqslant \varepsilon(k)\|\tau\|_{L^2} + Ck^{-1} \int_{|z|<k^{1/5}} |z|^4 |\tau(z)| e^{-|z|^2}\, dV \\ &\leqslant \varepsilon(k)\|\tau\|_{L^2} + Ck^{-1}\|\tau\|_{L^2}, \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last step, once again using the fact that on the set $\{|z| < k^{1/5}\}$ the metrics h and $2\pi k\omega$ are uniformly equivalent to $e^{-|z|^2}$ and the Euclidean metric respectively. $\square$

Finally we want to compute the L^2 norm of the section σ.

Lemma 6.9. *For large k we have*

$$\|\sigma\|^2_{L^2} = 1 - \frac{S_\omega(x)}{4\pi} k^{-1} + O(k^{-2}),$$

where S_ω is the scalar curvature of ω.

Proof. By the same arguments as in the previous Lemma, up to an error of $\varepsilon(k)$, which we can ignore, it is enough to compute the L^2-norm of s^k on the ball $\{|z| < k^{1/5}\}$. I.e. we need to compute

$$\int_{|z|<k^{1/5}} e^{-\Phi(z)} \frac{1}{(2\pi)^n n!} (\sqrt{-1}\partial\overline{\partial}\Phi)^n. \tag{61}$$

From (56) we have the expansion

$$\frac{1}{n!}(\sqrt{-1}\partial\overline{\partial}\Phi)^n = (1 - k^{-1} R_{k\bar{l}} z^k \bar{z}^l + k^{-3/2} q(z) + O(k^{-2}|z|^4))dV,$$

where $q(z)$ is a cubic polynomial in $z^i, \bar{z}^i$, and

$$dV = (\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \ldots \wedge dz^n \wedge d\bar{z}^n$$

is 2^n times the Euclidean volume form on $\mathbf{C}^n$. In addition we have

$$e^{-\Phi(z)} = e^{-|z|^2}\left(1 + \frac{k^{-1}}{4} R_{i\bar{j}k\bar{l}} z^i \bar{z}^j z^k \bar{z}^l - k^{-3/2} Q(z) + O(k^{-2}|z|^6)\right).$$

Extending the integral (61) over all of $\mathbf{C}^n$ introduces an error of $\varepsilon(k)$, so we have

$$\begin{aligned}(2\pi)^n \|\sigma\|^2_{L^2} = &\int_{\mathbf{C}^n} e^{-|z|^2}\, dV \\ &+ k^{-1} \int_{\mathbf{C}^n} e^{-|z|^2} \left(\frac{1}{4} R_{i\bar{j}k\bar{l}} z^i \bar{z}^j z^k \bar{z}^l - R_{k\bar{l}} z^k \bar{z}^l\right) dV \\ &+ k^{-3/2} \int_{\mathbf{C}^n} e^{-|z|^2} \left[Q(z) - q(z)\right] dV \\ &+ O(k^{-2}).\end{aligned} \tag{62}$$

These integrals can be computed by integrating in each coordinate direction separately and using the following formulas. First we have the 1-dimensional integral

$$\int_{\mathbf{C}} e^{-t|z|^2} \sqrt{-1} dz \wedge d\bar{z} = \frac{2\pi}{t}.$$

Differentiating this with respect to t we obtain

$$\int_{\mathbf{C}} |z|^2 e^{-|z|^2} \sqrt{-1} dz \wedge d\bar{z} = 2\pi, \qquad \int_{\mathbf{C}} |z|^4 e^{-|z|^2} \sqrt{-1} dz \wedge d\bar{z} = 4\pi.$$

All other integrals, where the number of z and $\bar{z}$ factors are not equal, will vanish by the mean value theorem. This implies that

$$\begin{aligned}\int_{\mathbf{C}^n} e^{-|z|^2} R_{i\bar{j}k\bar{l}} z^i \bar{z}^j z^k \bar{z}^l \, dV &= \sum_{i=1}^n (2\pi)^{n-1}(4\pi) R_{i\bar{i}i\bar{i}} + \sum_{i\neq j} (2\pi)^n (R_{i\bar{i}j\bar{j}} + R_{i\bar{j}j\bar{i}}) \\ &= 2(2\pi)^n \sum_{i,j} R_{i\bar{i}j\bar{j}} \\ &= 2(2\pi)^n S_{2\pi\omega}(x),\end{aligned}$$

where $S_{2\pi\omega}$ is the scalar curvature of the metric $2\pi\omega$. Similarly,

$$\int_{\mathbf{C}^n} e^{-|z|^2} R_{k\bar{l}} z^k \bar{z}^l \, dV = \sum_k (2\pi)^n R_{k\bar{k}} = (2\pi)^n S_{2\pi\omega}(x).$$

The integral involving Q, q vanishes. From (62) we therefore obtain

$$\|\sigma\|_{L^2}^2 = 1 - \frac{S_{2\pi\omega}(x)}{2} k^{-1} + O(k^{-2}).$$

Finally note that $S_{2\pi\omega} = \frac{1}{2\pi} S_\omega$. $\square$

We are now ready to complete the proof of the asymptotic expansion (55). Recall that $|\sigma(x)|_{h^k} = 1 + \varepsilon(k)$, so in particular $\sigma(x)$ does not vanish at x for large k. Let $E_x \subset H^0(M, L^k)$ be the space of sections vanishing at x, and let

$$\sigma = \eta + \tau$$

be the orthogonal decomposition of σ with $\eta \perp E_x$ and $\tau \in E_x$. Then

$$\|\eta\|_{L^2}^2 = \|\sigma\|_{L^2}^2 - \|\tau\|_{L^2}^2.$$

Since

$$\langle \tau, \tau \rangle_{L^2} = \langle \tau, \sigma \rangle_{L^2} \leqslant C k^{-1} \|\tau\|_{L^2},$$

we have $\|\tau\|_{L^2} \leqslant C k^{-1}$ from which it follows that

$$\|\eta\|_{L^2}^2 = \|\sigma\|_{L^2}^2 + O(k^{-2}) = 1 - \frac{S_\omega(x)}{4\pi} k^{-1} + O(k^{-2}).$$

Since η is orthogonal to every section vanishing at x, and $|\eta(x)|^2_{h^k} = 1+\varepsilon(k)$, the Bergman kernel at x is given by

$$B_{h^k}(x) = \frac{|\eta(x)|^2_{h^k}}{\|\eta\|^2_{L^2}} = 1 + \frac{S_\omega(x)}{4\pi}k^{-1} + O(k^{-2}).$$

This completes the proof of (55). In order to obtain stronger results, in particular the fact that the expansion holds in C^l norms, not just pointwise, one needs to work harder, but it is possible to argue along similar lines (see Tian [49] or Ruan [40]). An alternative approach is to use Fourier analytic techniques, as in Zelditch [57].

6.3 The algebraic and geometric Futaki invariants

Suppose that (M, L) is a polarized variety, with a $\mathbf{C}^*$-action λ (acting on both M and L). Let $\omega \in c_1(L)$ be a metric invariant under the S^1 subgroup. In this situation we can define the Futaki invariant differential-geometrically for the vector field generating the S^1-action as in Equation 40, and also algebraically as in Definition 5.24. We will use the Bergman kernel expansion to show that these two definitions are the same, up to a constant factor.

Let us write A_k for the infinitesimal generator of the $\mathbf{C}^*$-action λ on $H^0(M, L^k)$. By this we mean that the action is given by $t \mapsto t^{A_k}$, and so A_k has integral eigenvalues, which are the weights of the action. Recall that the Donaldson-Futaki invariant is defined by looking at the asymptotic behaviors

$$\begin{aligned} \dim H^0(M, L^k) &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}) \\ \mathrm{Tr}(A_k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}) \end{aligned}$$

for large k. We defined

$$DF(\lambda) = \frac{a_1}{a_0} b_0 - b_1.$$

On the differential geometric side, suppose that the vector field v generates the S^1-action on M, normalized so that the time 1 map generated by v is the identity. The fact that we have a lifting of the $\mathbf{C}^*$-action to L means that we have a Hamiltonian H for the vector field v, with respect to ω (we will see this in the proof below). The Futaki invariant of v is then

$$F(v) = \int_M H(S(\omega) - \hat{S})\,\frac{\omega^n}{n!}.$$

We are using a slightly different definition from before, but they only differ by a constant factor. We will prove the following.

Proposition 6.10. *In the situation above, we have $F(v) = 4\pi DF(\lambda)$.*

Proof. We need to compute the coefficients a_0, a_1, b_0, b_1 differential geometrically. For a_0, a_1 we have done this in Corollary 6.6, from which we have

$$a_0 = \int_M \frac{\omega^n}{n!}, \quad a_1 = \frac{1}{4\pi}\int_M S(\omega)\,\frac{\omega^n}{n!}.$$

To deal with b_0, b_1 in a similar way, one can use the equivariant Riemann-Roch formula (see Donaldson [16]). Instead, we will use the approximation to ω given by Corollary 6.5. Namely let $\{s_i\}$ be an orthonormal basis of $H^0(M, L^k)$ as in the construction of the Bergman kernel, and let $\varphi_k : M \to \mathbf{CP}^{N_k}$ be given by

$$\varphi_k(x) = [s_0(x) : \ldots : s_{N_k}(x)].$$

Writing

$$\omega_k = \frac{1}{2\pi k}\varphi_k^*\omega_{FS},$$

Corollary 6.5 says that $\omega - \omega_k = O(k^{-2})$ in any C^l norm. Moreover since ω is S^1-invariant, the metric ω_k will also be S^1-invariant.

Let us write $\varphi_k(x) = [\mathbf{x}]$, where $\mathbf{x} \in \mathbf{C}^{N_k+1}$ is a lift of $\varphi_k(x)$. We can then check that if we write $e^{2\pi\sqrt{-1}t} \in S^1$ for some $t \in \mathbf{R}$, then

$$\varphi_k(e^{2\pi\sqrt{-1}t}x) = [e^{2\pi\sqrt{-1}tA_k}\mathbf{x}].$$

The vector field v is therefore induced by the skew Hermitian matrix $2\pi\sqrt{-1}A_k$ on $\mathbf{CP}^{N_k}$. A Hamiltonian for this vector field with respect to ω_{FS} is given by (see Example 5.5)

$$H_k = \frac{-2\pi\sum_{i,j}(A_k)_{ij}Z_i\overline{Z}_j}{|Z|^2},$$

which in terms of the sections s_i is

$$H_k = \frac{-2\pi\sum_{i,j}(A_k)_{ij}\langle s_i, s_j\rangle_{h^k}}{B_{h^k}}.$$

Since $\omega - \omega_k = O(k^{-2})$, we can choose a Hamiltonian H for v with respect to ω, such that

$$H - \frac{1}{2\pi k}H_k = O(k^{-2}).$$

Then

$$\begin{aligned}\int_M HB_{h^k}\,\frac{\omega^n}{n!} &= \frac{1}{2\pi k}\int_M H_k B_{h^k}\,\frac{\omega^n}{n!} + O(k^{-2})\\ &= -\frac{1}{k^{n+1}}\sum_{i,j}(A_k)_{ij}\int_M \langle s_i, s_j\rangle_{h^k}\,\frac{(k\omega)^n}{n!} + O(k^{-2})\\ &= -\frac{1}{k^{n+1}}\mathrm{Tr}(A_k) + O(k^{-2}),\end{aligned}$$

where we used that the s_i are orthonormal. Using the expansion (55) for B_{h^k} we have

$$\mathrm{Tr}(A_k) = -k^{n+1}\int_M H\,\frac{\omega^n}{n!} - \frac{k^n}{4\pi}\int_M HS(\omega)\,\frac{\omega^n}{n!} + O(k^{n-1}).$$

It follows that

$$b_0 = -\int_M H\,\frac{\omega^n}{n!}, \quad b_1 = -\frac{1}{4\pi}\int_M HS(\omega)\,\frac{\omega^n}{n!}.$$

Using these in the definition of $DF(\lambda)$, we get

$$DF(\lambda) = -\frac{\hat{S}}{4\pi}\int_M H\,\frac{\omega^n}{n!} + \frac{1}{4\pi}\int_M HS(\omega)\,\frac{\omega^n}{n!} = \frac{1}{4\pi}F(v).$$

□

6.4 Lower bounds on the Calabi functional

Our goal in this section is to explain the proof of Donaldson's theorem [17], giving lower bounds for the Calabi functional in terms of Futaki invariants of test-configurations. Rather than reproducing all of the details from [17], we will focus on the main ideas.

Suppose that (X, L) is a polarized manifold. Recall that a test-configuration for (X, L) (of exponent 1 for simplicity) consists of an embedding $X \subset \mathbf{CP}^N$ using a basis of sections of L, and a $\mathbf{C}^*$-action $\lambda : \mathbf{C}^* \hookrightarrow GL(N+1, \mathbf{C})$. The flat limit

$$X_0 = \lim_{t\to 0}\lambda(t)\cdot X$$

is a projective scheme fixed by the action λ. In Definition 5.24 we defined the Donaldson-Futaki invariant $F(X_0, \lambda)$, and the norm $\|\lambda\|$. Our goal is to explain the proof of the following theorem.

Theorem 6.11. *If $\omega \in c_1(L)$ is a Kähler metric on X, then*

$$\|\lambda\| \cdot \|S(\omega) - \hat{S}\|_{L^2} \geqslant -4\pi F(X_0, \lambda).$$

In particular if X admits a cscK metric in $c_1(L)$, then $F(X_0, \lambda) \geqslant 0$ for any test-configuration.

Note that by replacing L by L^r one can obtain similar statements for any test-configuration, not just those of exponent 1, so the conclusion $F(X_0, \lambda) \geqslant 0$ really holds for any test-configuration.

6.4.1 Using the Bergman kernel

Suppose that we have a projective manifold $V \subset \mathbf{CP}^N$ of dimension n. Define the matrix $M(V)$ to be

$$M(V)_{ij} = \int_V \frac{Z^i \overline{Z}^j}{|Z|^2} \frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!},$$

and let $\underline{M}(V)$ be the trace free part of $M(V)$, i.e.

$$\underline{M}(V)_{ij} = M(V)_{ij} - \frac{\mathrm{Vol}(V)}{N+1}\delta_{ij}.$$

This is a moment map for the action of $SU(N+1)$ on the space of projective submanifolds of dimension n in $\mathbf{CP}^N$. The basic idea is, that in some sense as $N \to \infty$, the moment map $\underline{M}$ approaches the infinite dimensional moment map given by the scalar curvature, the link between the two being provided by the Bergman kernel expansion. We will now make this more precise.

Suppose that L is an ample line bundle on X, and let $\omega \in c_1(L)$.

Proposition 6.12. *There is a sequence of embeddings $M \to V_k \subset \mathbf{CP}^{N_k}$ using sections of L^k, such that*

$$\|\underline{M}(V_k)\| \leqslant \frac{k^{n/2-1}}{4\pi}\|S(\omega) - \hat{S}\|_{L^2} + O(k^{n/2-2}),$$

where $\|M\|^2 = \mathrm{Tr}(M^2)$ for any Hermitian matrix M.

Proof. As in the construction of the Bergman kernel, let $\{s_i\}$ be an orthonormal basis of $H^0(X, L^k)$, where the inner product on sections is defined using a metric h on L, whose curvature form is $2\pi\omega$. We let $V_k \subset \mathbf{CP}^{N_k}$ be the image of X under the embedding

$$\varphi_k : X \to \mathbf{CP}^{N_k},$$

given by this basis for large k. By applying a unitary transformation we can assume that $M(V_k)$ is diagonal, and so

$$M(V_k)_{ii} = \int_{V_k} \frac{|Z^i|^2}{|Z|^2} \frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} = \int_X |s_i|^2_{h^k} B_{h^k} \frac{(\frac{1}{2\pi}\varphi_k^*\omega_{FS})^n}{n!},$$

where B_{h^k} is the Bergman kernel. From Corollary 6.5 we know that

$$(\frac{1}{2\pi}\varphi_k^*\omega_{FS})^n = (k\omega)^n\Big(1 + O(k^{-2})\Big),$$

and also

$$B_{h_k} = 1 + \frac{S(\omega)}{4\pi}k^{-1} + O(k^{-2}).$$

It follows that

$$\begin{aligned} M(V_k)_{ii} &= \int_X |s_i|^2_{h^k}\left(1 - \frac{S(\omega)}{4\pi}k^{-1}\right)\frac{(k\omega)^n}{n!} + O(k^{-2}) \\ &= 1 - \frac{k^{-1}}{4\pi}\int_X |s_i|^2_{h^k} S(\omega)\frac{(k\omega)^n}{n!} + O(k^{-2}). \end{aligned} \tag{63}$$

The rank of the matrix $M(V_k)$ is

$$N_k + 1 = \dim H^0(X, L^k) = \int_X \frac{(k\omega)^n}{n!} + O(k^{-1}),$$

and the trace is

$$\begin{aligned} \sum_{i=0}^{N_k} M(V_k)_{ii} &= N_k + 1 - \frac{k^{-1}}{4\pi}\int_X B_{h^k} S(\omega)\frac{(k\omega)^n}{n!} + O(k^{n-2}) \\ &= N_k + 1 - \frac{k^{-1}}{4\pi}\int_X S(\omega)\frac{(k\omega)^n}{n!} + O(k^{n-2}). \end{aligned}$$

It follows that

$$\frac{\mathrm{Tr}(M(V_k))}{N_k + 1} = 1 - \frac{k^{-1}}{4\pi}\hat{S} + O(k^{-2}),$$

and so using (63) the trace free part of $M(V_k)$ is

$$\underline{M}(V_k)_{ii} = \frac{k^{-1}}{4\pi}\int_X |s_i|^2_{h^k}(\hat{S} - S(\omega))\,\frac{(k\omega)^n}{n!} + O(k^{-2}),$$

where we also used that $\|s_i\|_{L^2} = 1$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\underline{M}(V_k)_{ii}|^2 &\leqslant \frac{k^{-2}}{16\pi^2}\int_X |s_i|^2\,\frac{(k\omega)^n}{n!}\int_X |s_i|^2(\hat{S}-S(\omega))^2\,\frac{(k\omega)^n}{n!} + O(k^{-3})\\ &= \frac{k^{-2}}{16\pi^2}\int_X |s_i|^2(\hat{S}-S(\omega))^2\,\frac{(k\omega)^n}{n!} + O(k^{-3}),\end{aligned}$$

and so summing over the $N_k + 1 \sim k^n$ terms, we have

$$\begin{aligned}\|\underline{M}(V_k)\|^2 &\leqslant \frac{k^{-2}}{16\pi^2}\int_X B_{h^k}\,(\hat{S}-S(\omega))^2\,\frac{(k\omega)^n}{n!} + O(k^{n-3})\\ &= \frac{k^{n-2}}{16\pi^2}\int_X (\hat{S}-S(\omega))^2\,\frac{\omega^n}{n!} + O(k^{n-3}).\end{aligned}$$

Taking square roots gives the result we want. □

6.4.2 Lower bounds on $\|\underline{M}(V)\|$

Suppose that $V \subset \mathbf{CP}^N$ is a projective manifold, and let $\lambda : \mathbf{C}^* \hookrightarrow GL(N+1,\mathbf{C})$ be a one-parameter subgroup. We further require now that the image of the unit complex numbers $S^1 \subset \mathbf{C}^*$ under λ lies in $U(N+1)$. We will give a lower bound for $\|\underline{M}(V)\|$, which is a finite dimensional analog of Theorem 6.11.

For any $t \in \mathbf{C}^*$, let us write $V_t = \lambda(t)\cdot V$. Suppose that $\lambda(t) = t^A$ for a Hermitian matrix A with integer eigenvalues. Then the S^1-action on $\mathbf{CP}^N$ is induced by the skew Hermitian matrix $-\sqrt{-1}A$ (we think of A as acting on $\mathbf{CP}^N$ by the dual action in order to fit in with our convention for the Futaki invariant). A Hamiltonian function for the vector field generating this circle action is then given by

$$h = -\frac{A_{ij}Z^i\overline{Z}^j}{|Z|^2}.$$

Define the function

$$f(t) = -\mathrm{Tr}(\underline{A}\underline{M}(V^t)) = -\mathrm{Tr}(\underline{A}M(V^t)),$$

where $\underline{A}$ is the trace free part of A. Then

$$f(t) = \int_{V_t} h\,\frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} + \frac{\mathrm{Tr}(A)}{N+1}\mathrm{Vol(V)}.$$

The key point is that $f(t)$ is non-decreasing for $t \in \mathbf{R}_{>0}$.

Lemma 6.13. *Restricting $f(t)$ to real numbers $t > 0$, we have $f'(t) \geqslant 0$.*

Proof. This is essentially the calculation (53), using the fact that $\underline{M}$ is a moment map. Nevertheless we can check it directly. Let v_h be the vector field generating the S^1-action on $\mathbf{CP}^N$. Then

$$Jv_h = -\operatorname{grad} h.$$

Let us write $\Phi_s : \mathbf{CP}^N \to \mathbf{CP}^N$ for the 1-parameter group of diffeomorphisms generated by Jv_h (this corresponds to approaching 0 along the positive real axis in $\mathbf{C}^*$). It is enough to compute the following derivative at $s = 0$.

$$\begin{aligned}\frac{d}{ds}\bigg|_{s=0} \int_{\Phi_s(V)} h \frac{\omega_{FS}^n}{n!} &= \frac{d}{ds}\bigg|_{s=0} \int_V \Phi_s^*(h) \frac{(\Phi_s^*\omega_{FS})^n}{n!} \\ &= \int_V (Jv_h)(h) \frac{\omega_{FS}^n}{n!} + \int_V h \frac{nL_{Jv_h}\omega_{FS} \wedge \omega_{FS}^{n-1}}{n!}\end{aligned}$$

Since $(-\operatorname{grad} h)(h) = -|\operatorname{grad} h|^2$, the first term is

$$\int_V -|\operatorname{grad} h|^2 \frac{\omega_{FS}^n}{n!}.$$

For the second term, recall from (50) that $L_{Jv_h}\omega_{FS} = -2\sqrt{-1}\partial\bar{\partial}h$. Integrating by parts we have

$$\int_V h\, 2n\sqrt{-1}\partial\bar{\partial}h \wedge \omega_{FS}^{n-1} = -\int_V 2n\partial h \wedge \bar{\partial}h \wedge \omega_{FS}^{n-1} = \int_V 2|\partial h|_V^2\, \omega_{FS}^n,$$

where we used Lemma 4.6, and we want to emphasize that $|\partial h|_V^2$ is the norm of only the part of ∂h which is tangential to V. In terms of the real gradient, $|\partial h|_V^2 = \frac{1}{2}|\operatorname{grad} h|_V^2$, where again only the tangential part is considered. It follows that

$$\frac{d}{ds}\bigg|_{s=0} \int_{\Phi_s(V)} h \frac{\omega_{FS}^n}{n!} = -\int_V |\operatorname{grad} h|_N^2 \frac{\omega_{FS}^n}{n!} \leqslant 0,$$

where $|\operatorname{grad} h|_N^2$ means that we are taking the norm of the normal component to V. Increasing t corresponds to flowing along $-Jv_h$, so the result that we want follows. □

Let us write $V_0 = \lim_{t\to 0} V_t$ for the flat limit. A crucial fact is that there is an algebraic cycle $|V_0|$ associated to V_0, which can be thought of as the union of the n dimensional irreducible components of V_0, counted with multiplicities. In this way one can make sense of integrals over V_0, and if we define

$$FCh(A, V_0) = -\int_{|V_0|} h\,\frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} - \frac{\mathrm{Tr}(A)}{N+1}\mathrm{Vol}(V),$$

then

$$\lim_{t\to 0} f(t) = -FCh(A, V_0),$$

since the convergence $V_t \to |V_0|$ holds in the sense of currents. The monotonicity of $f(t)$ implies the following, which is the finite dimensional analog of Theorem 6.11.

Proposition 6.14. *We have*

$$\|\underline{A}\| \cdot \|\underline{M}(V)\| \geqslant -FCh(A, V_0).$$

Proof. The monotonicity of f implies that

$$-\mathrm{Tr}(\underline{A}\underline{M}(V)) = f(1) \geqslant \lim_{t\to 0} f(t) = -FCh(A, V_0).$$

The result then follows from the Cauchy-Schwarz inequality. □

In order to relate this to the Futaki invariant, we need to be able to compute $FCh(A, V_0)$ algebro-geometrically. Recall from Section 5.7, that given the $\mathbf{C}^*$-action λ, there is an induced (dual) $\mathbf{C}^*$-action on the homogeneous coordinate ring

$$R = \mathbf{C}[x_0, \ldots, x_N]/I_0,$$

where I_0 is the homogeneous ideal corresponding to the flat limit V_0. Let us write A_k for the generator of the $\mathbf{C}^*$-action on R_k, so the total weight of the action if $w_k = \mathrm{Tr}(A_k)$. Note that if $V \subset \mathbf{CP}^N$ is not contained in any hyperplane, then R_1 consists of all the linear polynomials, and $A_1 = A$. We need the following.

Lemma 6.15. *For large k we have*

$$\mathrm{Tr}(A_k) = -k^{n+1}\int_{|V_0|} h\,\frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} + O(k^n). \tag{64}$$

If V_0 were smooth, then this follows from the arguments that we used to prove Proposition 6.10. In general, more involved arguments are required. One approach is to reduce the problem to the case when A has constant weights, and so h is a constant function. Then one needs to relate the volume of V_0 to the leading order term of its Hilbert polynomial (see Donaldson [17] for this approach). An alternative approach, following Wang [54] is to degenerate V_0 into a union of linear subspaces (with multiplicities) in such a way that the two sides of (64) remain unchanged in the limit, and then check directly that the equation holds for linear subspaces.

6.4.3 Putting the pieces together

We would now like to combine the results of the previous two sections to complete the proof of Theorem 6.11. We start with a test-configuration λ for (X, L) of exponent 1, and a metric $\omega \in c_1(L)$. We would like to apply Proposition 6.14 to the sequence $V_k \subset \mathbf{CP}^{N_k}$ obtained in Proposition 6.12. For this we need to use λ to define $\mathbf{C}^*$-actions

$$\lambda_k : \mathbf{C}^* \hookrightarrow GL(N_k + 1, \mathbf{C}),$$

in such a way that λ_k maps $S^1 \subset \mathbf{C}^*$ into $U(N_k + 1)$. A natural way to do this is to work with filtrations instead of $\mathbf{C}^*$-actions.

Let us write $S = \mathbf{C}[x_0, \ldots, x_N]$ for the polynomial ring in $N + 1 = \dim H^0(X, L)$ variables. The homogeneous coordinate ring of X is given by

$$R = S/I$$

for a homogeneous ideal I. The $\mathbf{C}^*$-action λ induces a filtration (in fact even a grading) on S, defined by

$$F_i S = \{\text{span of elements } f \in S \text{ with weights } \geqslant i\},$$

and this descends to a filtration $F_i R$ on R, such that

$$\ldots \supset F_i R \supset F_{i+1} R \supset \ldots.$$

We can further restrict the filtration to the degree k piece R_k for any k. The key observation is that all the data from the test-configuration that we need can be extracted from the filtration. In particular any two $\mathbf{C}^*$-actions which induce the same filtration will have the same numerical invariants.

Lemma 6.16. *We have the following*

$$\begin{aligned}
N_k + 1 = \dim R_k &= \sum_i (\dim F_i R_k - \dim F_{i+1} R_k) \\
\mathrm{Tr}(A_k) &= \sum_i i(\dim F_i R_k - \dim F_{i+1} R_k) \\
\mathrm{Tr}(A_k^2) &= \sum_i i^2(\dim F_i R_k - \dim F_{i+1} R_k).
\end{aligned}$$

Note that there are only finitely many non-zero terms in each sum, since for fixed k, the filtration F_iR_k must stabilize. Also, A_k denotes the generator of the $\mathbf{C}^*$-action on $H^0(X_0, L^k)$ as in the definition of the Donaldson-Futaki invariant.

Now for any k, recall that the embedding $X \to V_k \subset \mathbf{CP}^{N_k}$ was given by an orthonormal basis, for a suitable choice of Hermitian metric on $R_k = H^0(X, L^k)$. Using this metric we can decompose R_k into an orthogonal direct sum

$$R_k = \ldots \oplus F_{i-1}R_k/F_iR_k \oplus F_iR_k/F_{i+1}R_k \oplus \ldots,$$

where only finitely many terms are non-zero. We then define the $\mathbf{C}^*$-action λ_k to act with weight i on the summand $F_iR_k/F_{i+1}R_k$. In this way, S^1 will act by unitary transformations of R_k. Applying Proposition 6.14 to this action, we get

$$\|\underline{A}_k\| \cdot \|\underline{M}(V_k)\| \geqslant \int_{(V_k)_0} h_k \frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} + \frac{\mathrm{Tr}(A_k)}{N_k+1}\mathrm{Vol}(V_k), \tag{65}$$

with self explanatory notation. From Lemma 6.15 we have for fixed k, as $l \to \infty$

$$\mathrm{Tr}(A_{kl}) = -l^{n+1}\int_{(V_k)_0} h_k \frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} + O(l^n).$$

Comparing this to the expansion (as $k \to \infty$)

$$\mathrm{Tr}(A_k) = b_0k^{n+1} + O(k^n),$$

we see that

$$\int_{(V_k)_0} h_k \frac{(\frac{1}{2\pi}\omega_{FS})^n}{n!} = -b_0k^{n+1}.$$

From (65) we therefore have

$$\begin{aligned}(\mathrm{Tr}(\underline{A}_k^2))^{1/2} \cdot \|\underline{M}(V_k)\| &\geqslant -b_0k^{n+1} + \frac{b_0k^{n+1} + b_1k^n + O(k^{n-1})}{a_0k^n + a_1k^{n-1} + O(k^{n-2})}a_0k^n \\ &= -b_0k^{n+1} + (b_0k^{n+1} + b_1k^n + O(k^{n-1}))(1 - \frac{a_1}{a_0}k^{-1} + O(k^{-2})) \\ &= k^n\left(b_1 - \frac{a_1}{a_0}b_0\right) + O(k^{n-1}).\end{aligned}$$

Combining this with Proposition 6.12, and the definitions of the Donaldson-Futaki invariant $DF(X_0, \lambda)$ and the norm $\|\lambda\|$, we have

$$\Big(\|\lambda\| k^{n/2+1} + O(k^{n/2})\Big) \left(\frac{k^{n/2-1}}{4\pi} \|S(\omega) - \hat{S}\|_{L^2} + O(k^{n/2-2}) \right) \geqslant -k^n DF(X_0, \lambda) + O(k^{n-1}).$$

Letting $k \to \infty$ we find

$$\|\lambda\| \cdot \|S(\omega) - \hat{S}\|_{L^2} \geqslant -4\pi DF(X_0, \lambda),$$

which is the statement of Theorem 6.11.

The following corollary is immediate from Theorem 6.11.

Corollary 6.17. *Suppose that X admits a cscK metric $\omega \in c_1(L)$. Then (X, L) is K-semistable.*

6.5 Exercises

Exercise 6.1. Let L be a positive line bundle over a compact Kähler manifold (M, ω) of dimension n. Choose a metric h on L with positive curvature (not related to ω!). On $H^0(M, L^k)$, define the inner product

$$\langle s, t \rangle_{L^2} = \int_M \langle s, t \rangle_{h^k} \frac{\omega^n}{n!}.$$

Given any orthonormal basis $\{s_0, \ldots, s_{N_k}\}$ for this inner product, define the Bergman kernel

$$B_k(x) = \sum_{i=0}^{N_k} |s_i(x)|^2_{h^k}.$$

What are the first two terms in the asymptotic expansion of $B_k(x)$ as $k \to \infty$?

7 CscK metrics on blowups

Suppose that M is a compact Kähler manifold with a cscK metric ω. In this section we will describe how one can construct cscK metrics on the blowup of M at a point. We will only discuss the simplest setting, when M has no holomorphic vector fields, and the dimension $n > 2$. More general results can be found in Arezzo-Pacard [2, 3], Arezzo-Pacard-Singer [4], and also [46].

7.1 The basic strategy

The technique used for constructing cscK metrics on the blowup at a point is very general, used in a wide variety of problems in geometric analysis. There are two main steps in the argument. Starting with a cscK metric ω on M and a point $p \in M$, one first constructs a family of metrics ω_ε on the blowup $\mathrm{Bl}_p M$, depending on a small parameter $\varepsilon > 0$. The metrics ω_ε are obtained by modifying ω on a very small neighborhood of p, and as $\varepsilon \to 0$, the metrics ω_ε converge to ω in a suitable sense (away from the point p). The second step is to perturb ω_ε in its Kähler class to obtain a cscK metric. This involves studying the linearization of the scalar curvature operator, and it will only be possible for sufficiently small ε.

For the result that we prove here, we can get away with a fairly crude construction of the approximate solutions ω_ε. To obtain more refined results such as in [3, 4, 46], one needs to build the approximate solutions more carefully.

7.1.1 Blowups

Let us recall briefly how to construct the blowup of a Kähler manifold M at a point p. The blowup $\mathrm{Bl}_0\mathbf{C}^n$ of $\mathbf{C}^n$ at the origin is simply the total space of the $\mathcal{O}(-1)$ bundle over $\mathbf{CP}^{n-1}$. Let us write

$$E \cong \mathbf{CP}^{n-1} \subset \mathrm{Bl}_0\mathbf{C}^n$$

for the zero section. There is a holomorphic map

$$\begin{aligned} \pi : \mathcal{O}(-1) &\to \mathbf{C}^n \\ \Big([z_0 : \ldots : z_n], (z_0, \ldots, z_n)\Big) &\mapsto (z_0, \ldots, z_n), \end{aligned}$$

where recall that the fiber of $\mathcal{O}(-1)$ over the point $[z_0 : \ldots : z_n]$ is simply the line in $\mathbf{C}^{n+1}$ spanned by $(z_0, \ldots, z_n)$. The map π restricts to a

biholomorphism
$$\pi : \mathrm{Bl}_0\mathbf{C}^n \setminus E \to \mathbf{C}^n \setminus \{0\}.$$

Suppose now that M is a complex manifold of dimension $n > 1$, and $p \in M$. We can identify a neighborhood of p with a ball $B \subset \mathbf{C}^n$, such that p corresponds to the origin. The blowup Bl_pM is then constructed by replacing $B \subset M$ by $\pi^{-1}(B) \subset \mathrm{Bl}_0\mathbf{C}^n$, using the biholomorphism

$$\pi : \pi^{-1}(B \setminus \{0\}) \to B \setminus \{0\}.$$

The result is a complex manifold Bl_pM, equipped with a holomorphic map

$$\pi : \mathrm{Bl}_pM \to M,$$

called the blowdown map. The preimage $E = \pi^{-1}(p)$ is a copy of $\mathbf{CP}^{n-1}$, and π restricts to a biholomorphism

$$\pi : \mathrm{Bl}_pM \setminus E \xrightarrow{\sim} M \setminus \{p\}.$$

An application of the Mayer-Vietoris sequence shows that (if $n > 1$)

$$H^2(\mathrm{Bl}_pM, \mathbf{R}) \cong H^2(M, \mathbf{R}) \oplus \mathbf{R}[E],$$

where $[E]$ denotes the Poincaré dual of E. We will see that if ω is a Kähler metric on M, then for sufficiently small $\varepsilon > 0$, the class

$$\pi^*[\omega] - \varepsilon^2[E]$$

is a Kähler class on Bl_0M. Our goal is to prove the following theorem.

Theorem 7.1 (Arezzo-Pacard)**.** *Suppose that M is a compact Kähler manifold with no holomorphic vector fields, and ω is a cscK metric on M. Then for any $p \in M$ the blowup Bl_pM admits a cscK metric in the Kähler class*

$$\pi^*[\omega] - \varepsilon^2[E]$$

for sufficiently small $\varepsilon > 0$.

7.1.2 The Burns-Simanca metric

A basic ingredient in constructing cscK metrics on blowups is a scalar flat, asymptotically flat metric on $\mathrm{Bl}_0\mathbf{C}^n$. This metric was found by Burns and Simanca (see [24], [42]). For $n > 2$ the metric can be written as

$$\eta = \sqrt{-1}\partial\overline{\partial}\Big(|w|^2 + O(|w|^{4-2n})\Big) \tag{66}$$

as $|w| \to \infty$, in terms of the standard coordinates on $\mathrm{Bl}_0\mathbf{C}^n \setminus E \cong \mathbf{C}^n \setminus \{0\}$.

We will briefly describe the construction of the metric η here, using the methods we used in Section 4.4. We are trying to construct a metric on the total space of the $\mathcal{O}(-1)$-bundle over $\mathbf{CP}^{n-1}$. Choose a Hermitian metric h on $\mathcal{O}(-1)$ with curvature form $F(h) = -\omega_{FS}$. We will construct the metric in the form

$$\eta = \sqrt{-1}\partial\bar{\partial} f(s),$$

where f is a suitable strictly convex function, and $s = \log |z|_h^2$ is the log of the fiberwise norm. As in Section 4.4 we can use coordinates z on $\mathbf{CP}^{n-1}$ and a fiberwise coordinate w, so that $|(z,w)|_h^2 = |w|^2 h(z)$. We can choose coordinates at a point such that $dh = 0$. Then

$$\eta = f'(s) p^*\omega_{FS} + f''(s)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2},$$

where $p : \mathcal{O}(-1) \to \mathbf{CP}^{n-1}$ is the projection map. We get

$$\eta^n = \frac{f''(s)}{|w|^2}(f'(s))^{n-1} p^*\omega_{FS}^{n-1} \wedge \sqrt{-1}dw \wedge d\bar{w},$$

which is true at any point, not just where $dh = 0$. The Ricci form is therefore

$$\rho = -\sqrt{-1}\partial\bar{\partial}\log(f''(s)(f'(s))^{n-1}) + p^*(n\omega_{FS}),$$

using that $\mathrm{Ric}(\omega_{FS}) = n\omega_{FS}$. Taking the Legendre transform of f as in Section 4.4 we can rewrite this in terms of the function $\varphi(\tau) = f''(s)$, where $\tau = f'(s)$. We have

$$\begin{aligned}
\eta &= \tau p^*\omega_{FS} + \varphi(\tau)\frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2} \\
\rho &= -\sqrt{-1}\partial\bar{\partial}\log(\varphi\tau^{n-1}) + np^*\omega_{FS} \\
&= \left(-\varphi' - \frac{(n-1)\varphi}{\tau} + n\right) p^*\omega_{FS} - \varphi\left(\varphi' + \frac{(n-1)\varphi}{\tau}\right)' \frac{\sqrt{-1}dw \wedge d\bar{w}}{|w|^2}.
\end{aligned}$$

Taking the trace, we have

$$S(\eta) = \frac{-1}{\tau^2}\Big[\tau^2\varphi'' + 2\tau(n-1)\varphi' + (n-1)(n-2)\varphi - \tau n(n-1)\Big].$$

To obtain a metric which on the zero section restricts to ω_{FS}, we need to find φ defined on $[1, \infty)$, such that

$$\varphi(1) = 0, \quad \varphi'(1) = 1.$$

The equation $S(\eta) = 0$ is equivalent to

$$\frac{d^2}{d\tau^2}\Big[\tau^{n-1}\varphi\Big] = \tau^{n-2}n(n-1).$$

Integrating this twice, using the boundary conditions, we get

$$\varphi(\tau) = \tau - (n-1)\tau^{2-n} + (n-2)\tau^{1-n}.$$

We need to change variables back to s to see the asymptotics of the Kähler potential in complex coordinates. Note that

$$\tau = \frac{d}{ds}f(s) = \varphi\frac{d}{d\tau}f(s),$$

so

$$\frac{d}{d\tau}f(s) = \tau\varphi^{-1} = (1 - (n-1)\tau^{1-n} + (n-2)\tau^{-n})^{-1}.$$

For large τ we have

$$\frac{d}{d\tau}f(s) = 1 + (n-1)\tau^{1-n} - (n-2)\tau^{-n} + O(\tau^{2-2n}),$$

and so up to changing f by a constant,

$$f(s) = \tau - \frac{n-1}{n-2}\tau^{2-n} + \frac{n-2}{n-1}\tau^{1-n} + O(\tau^{3-2n}).$$

We also have

$$\frac{ds}{d\tau} = \varphi^{-1} = \tau^{-1}(1 + (n-1)\tau^{1-n} - (n-2)\tau^{-n} + O(\tau^{2-2n}))$$

for large τ, so up to adding a constant to s (which corresponds to scaling the metric h), we have

$$\log|z|_h^2 = s = \log\tau + O(\tau^{1-n}).$$

Using this,

$$f(s) = |z|_h^2 - \frac{n-1}{n-2}|z|_h^{4-2n} + \frac{n-2}{n-1}|z|_h^{2-2n} + O(|z|^{6-4n}).$$

Now recall that under the biholomorphism $\mathcal{O}(-1) \setminus \mathbf{CP}^{n-1} \cong \mathbf{C}^n \setminus \{0\}$, the metric h is given by a multiple of the Euclidean metric

$$|(z_1, \ldots, z_n)|_h^2 = c(|z_1|^2 + \ldots + |z_n|^2).$$

This shows that the metric η is of the form given in Equation (66).

7.1.3 The approximate solution

Let us suppose now that ω is a cscK metric on M, and we picked a point $p \in M$. In order to construct a metric on $\mathrm{Bl}_p M$ which has approximately constant scalar curvature, the idea is to replace the metric ω on a small neighbourhood of p with a suitably scaled down copy of η. To do this we use cut-off functions to patch together the Kähler potentials.

Suppose that z^i are normal coordinates centered at p, so that near p the metric ω is of the form

$$\omega = \sqrt{-1}\partial\overline{\partial}\Big(|z|^2 + \varphi_1(z)\Big),$$

where $\varphi_1(z) = O(|z|^4)$. For simplicity we can assume that the z^i are defined for $|z| < 1$. Fix a parameter ε, and let

$$r_\varepsilon = \varepsilon^{\frac{n-1}{n}}.$$

We will glue $\varepsilon^2\eta$ to ω, on the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$. Under the change of variables $z = \varepsilon w$ we have

$$\varepsilon^2\eta = \sqrt{-1}\partial\overline{\partial}\Big(|z|^2 + \varepsilon^2\varphi_2(\varepsilon^{-1}z)\Big),$$

where $\varphi_2(z) = O(|z|^{4-2n})$.

Let us choose a smooth function $\gamma : \mathbf{R} \to [0,1]$ such that

$$\gamma(x) = \begin{cases} 1 \text{ if } x \geqslant 2 \\ 0 \text{ if } x \leqslant 1, \end{cases}$$

and define $\gamma_1(z) = \gamma(|z|/r_\varepsilon)$. Also, let $\gamma_2 = 1 - \gamma_1$. Define the metric ω_ε on $M \setminus \{p\}$ by letting

$$\omega_\varepsilon = \begin{cases} \omega \text{ on } M \setminus B_{2r_\varepsilon} \\ \sqrt{-1}\partial\overline{\partial}\Big(|z|^2 + \gamma_1(z)\varphi_1(z) + \varepsilon^2\gamma_2(z)\varphi_2(\varepsilon^{-1}z)\Big) \text{ on } B_{2r_\varepsilon} \setminus B_{r_\varepsilon} \\ \varepsilon^2\eta \text{ on } B_{r_\varepsilon} \setminus \{p\}. \end{cases}$$

The reason for our choice of $r_\varepsilon = \varepsilon^{(n-1)/n}$ is that this way on the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$ we have

$$\gamma_1(z)\varphi_1(z) + \varepsilon^2\gamma_2(z)\varphi_2(\varepsilon^{-1}z) = O(|z|^4).$$

The metric ω_ε is positive definite everywhere if ε is sufficiently small. It also naturally extends to a metric on $\mathrm{Bl}_p M$ which we will write as ω_ε as well. Since the volume of the exceptional divisor E with this metric is $\frac{\varepsilon^{2n-2}}{(n-1)!}$ and we have not changed ω outside a small ball, the Kähler class of ω_ε is $\pi^*[\omega] - \varepsilon^2[E]$.

7.1.4 The equation

Our goal is to perturb ω_ε into a cscK metric for sufficiently small ε. This means that we need to find a smooth function φ on $\mathrm{Bl}_p M$ such that $\omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi$ is cscK. A small technical nuisance is caused by the fact that adding a constant to φ does not change the metric. One way to overcome this is to choose a point $q \in M$ outside the unit ball around p, and try to solve the equation

$$S(\omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi) - S(\omega) - \varphi(q) = 0. \tag{67}$$

This equation is no longer invariant under adding constants to φ.

We will solve the equation using the contraction mapping principle, and for this a crucial role is played by the linearization of the scalar curvature operator. At any metric ω this is given by Lemma 4.4 as

$$\begin{aligned} L_\omega(\varphi) := \left.\frac{d}{dt}\right|_{t=0} S(\omega + t\sqrt{-1}\partial\overline{\partial}\varphi) &= -\mathcal{D}^*\mathcal{D}\varphi + g^{j\bar{k}}\partial_j S(\omega)\partial_{\bar{k}}\varphi \\ &= -\Delta^2\varphi - R_\omega^{j\bar{k}}\partial_j\partial_{\bar{k}}\varphi, \end{aligned}$$

where $R_\omega^{j\bar{k}}$ is the Ricci curvature of ω with the indices raised. An important observation is that if $S(\omega)$ is constant, then

$$L_\omega(\varphi) = -\mathcal{D}^*\mathcal{D}\varphi,$$

and so if M is compact, the kernel of $L_\omega(\varphi)$ coincides with the kernel of $\mathcal{D}$. If there are no non-zero holomorphic vector fields on M, then the kernel of $\mathcal{D}$ consists of only the constants so in this case L_ω is an isomorphism when restricted to the L^2-orthogonal complement of the constants. Again one can remove the issue with the constant functions by considering the operator

$$\tilde{L}_\omega(\varphi) = L_\omega(\varphi) - \varphi(q),$$

where $q \in M$ is a point we fix in advance. It is then easy to check that if M is compact, ω is cscK, and M has no holomorphic vector fields, then

$$\tilde{L}_\omega : C^{k,\alpha}(M) \to C^{k-4,\alpha}(M)$$

is an isomorphism.

In order to solve Equation (67), the most important step is to show that the linearization $\tilde{L}_{\omega_\varepsilon}$ is invertible, and to obtain bounds on the norm of the inverse in suitable Banach spaces. It turns out that the right spaces to use are certain weighted Hölder spaces. In the next section we will discuss the basic theory of elliptic operators acting between weighted spaces.

7.2 Analysis in weighted spaces

7.2.1 The case of $\mathbf{R}^n$

We will first study some of the mapping properties of the Laplacian on $\mathbf{R}^n$. To define weighted spaces, let $\rho : \mathbf{R}^n \to \mathbf{R}$ be a smooth function such that

$$\rho(x) = \begin{cases} |x| \ \text{ for } |x| \geqslant 2 \\ 1 \ \text{ for } |x| \leqslant 1. \end{cases}$$

Let $f : \mathbf{R}^n \to \mathbf{R}$. For $\alpha \in (0,1)$, $\delta \in \mathbf{R}$ and a non-negative integer k, the weighted Hölder norm of f is defined to be

$$\begin{aligned} \|f\|_{C^{k,\alpha}_\delta(\mathbf{R}^n)} = \sum_{j\leqslant k} \sup_x \rho(x)^{-\delta+j} |\nabla^j f(x)| \\ + \sup_{x\neq y} \min\{\rho(x),\rho(y)\}^{-\delta+k+\alpha} \frac{|\nabla^k f(x) - \nabla^k f(y)|}{|x-y|^\alpha}. \end{aligned} \tag{68}$$

So roughly speaking $\|f\|_{C^{k,\alpha}_\delta} \leqslant C$ means that $\nabla^i f$ should decay (or grow at most) like $C\rho^{\delta-i}$ for $i \leqslant k$.

There is a more geometric way to think of these weighted norms, in terms of rescalings. For $r > 1$, define

$$\begin{gathered} f_r : B_2 \setminus B_1 \to \mathbf{R} \\ f_r(x) = r^{-\delta} f(rx), \end{gathered}$$

where B_r is the ball of radius r in $\mathbf{R}^n$. In other words f_r is the pullback of $r^{-\delta} f$ under the scaling map $B_2 \setminus B_1 \to B_{2r} \setminus B_r$. Then for any i we have

$$\nabla^i f_r(x) = r^{-\delta+i} \nabla^i f(rx).$$

On the annulus $B_{2r} \setminus B_r$, ρ is equal to r up to a factor of 2. Because of this, one can give an equivalent definition of the weighted norm by

$$\|f\|_{C^{k,\alpha}_\delta(\mathbf{R}^n)} = \|f\|_{C^{k,\alpha}(B_2)} + \sup_{r>1} \|f_r\|_{C^{k,\alpha}(B_2\setminus B_1)}, \tag{69}$$

in terms of the usual Hölder norms. This new definition will of course not be equal to (68), but it will define an equivalent norm.

We say that $f \in C^{k,\alpha}_\delta(\mathbf{R}^n)$, if the weighted norm of f is finite. One can show that these weighted Hölder spaces are Banach spaces, and the embedding $C^{k,\alpha}_\delta \subset C^{l,\beta}_\gamma$ is compact if $\delta < \gamma$ and $k+\alpha > l+\beta$. Note also that it follows from the definition that if $f \in C^{k,\alpha}_\delta$, then for $i \leqslant k$ we have $\nabla^i f \in C^{k-i,\alpha}_{\delta-i}$, and

$$\|\nabla^i f\|_{C^{k-i,\alpha}_{\delta-i}} \leqslant C\|f\|_{C^{k,\alpha}_\delta}$$

for some C independent of f. In particular for any δ the Laplacian defines a bounded linear map

$$\Delta_\delta : C^{k,\alpha}_\delta(\mathbf{R}^n) \to C^{k-2,\alpha}_{\delta-2}(\mathbf{R}^n).$$

Certain nonlinear operators also define bounded maps between suitable weighted spaces, thanks to the boundedness of the multiplication maps

$$\begin{aligned} C^{k,\alpha}_\delta \times C^{k,\alpha}_{\delta'} &\to C^{k,\alpha}_{\delta+\delta'} \\ (f,g) &\mapsto fg \end{aligned}$$

The basic question is to determine the mapping properties of Δ_δ. It turns out that except for a discrete set of δ, the Laplacian Δ_δ defines a Fredholm map.

Definition 7.2. We call $\delta \in \mathbf{R}$ an indicial root (of the Laplacian on $\mathbf{R}^n$), if there is a harmonic function $f : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ which grows at the rate $\rho(x)^\delta$. In other words we require that $f \in C^{2,\alpha}_\delta$, but $f \notin C^{2,\alpha}_{\delta'}$ for any $\delta' < \delta$. A direct calculation in polar coordinates shows that the set of indicial roots is

$$\{0,1,2,\ldots\} \cup \{2-n, 1-n, -n, \ldots\} = \mathbf{Z} \setminus (2-n, 0).$$

Indeed for every integer $k \geqslant 0$ there are harmonic polynomials of degree k, and one can check that if $P_k(x)$ is a degree k harmonic polynomial, then in addition to $\Delta P_k(x) = 0$ we also have

$$\Delta(|x|^{2-n-2k} P_k(x)) = 0,$$

so $2-n-2k+k = 2-n-k$ is also an indicial root.

The basic result that we need is the following.

Theorem 7.3. *If δ is not an indicial root, then*

$$\Delta_\delta : C^{k,\alpha}_\delta(\mathbf{R}^n) \to C^{k-2,\alpha}_{\delta-2}(\mathbf{R}^n)$$

is a Fredholm operator. Moreover

$$\mathrm{Im}(\Delta_\delta) = (\mathrm{Ker}(\Delta_{2-n-\delta}))^\perp,$$

where the orthogonal complement is taken with respect to the L^2-product.

We will not prove this theorem, but instead just verify a few special cases. Some references for the general theory are Lockhart-McOwen [29] or Pacard [37].

Theorem 7.4. *Suppose that $n > 2$ and $\delta \in (2-n, 0)$. Then the Laplacian*

$$\Delta_\delta : C^{k,\alpha}_\delta(\mathbf{R}^n) \to C^{k-2,\alpha}_{\delta-2}(\mathbf{R}^n)$$

is an isomorphism.

Sketch of proof. Recall that the fundamental solution of the Laplacian is

$$G(x) = \frac{1}{n(n-2)c_n}|x|^{2-n},$$

where c_n is the volume of the unit ball in $\mathbf{R}^n$. If $u \in C^{k-2,\alpha}_{\delta-2}$, then we can define $f : \mathbf{R}^n \to \mathbf{R}$ by

$$f(x) = \int_{\mathbf{R}^n} G(x-y)u(y)\,dy.$$

One can then check that f is well-defined, $\Delta f = u$, and

$$|f(x)| \leqslant C\|u\|_{C^{k-2,\alpha}_{\delta-2}}\rho(x)^\delta.$$

In order to obtain estimates for the derivatives of f, we can apply the Schauder estimate 2.7 to the rescaled functions f_r in (69) (the details are Exercise 7.1). This shows that Δ_δ is surjective. It is also injective, since from Liouville's theorem (Corollary 2.4) we know that any harmonic function which decays at infinity has to vanish. $\square$

In contrast to this, we can check that for $\delta = 0$, the image of Δ_δ is not closed, so the operator is not Fredholm. The basic idea is that if $f : \mathbf{R}^n \to \mathbf{R}$ is a function such that

$$f(x) = \log\log|x| \text{ for sufficiently large } x,$$

then on the one hand

$$\Delta f(x) \sim \frac{1}{|x|^2\log|x|} \text{ for large } x,$$

so $\Delta f(x) \in C^{k-2,\alpha}_{-2}$, but on the other hand $f \notin C^{k,\alpha}_0$ since f is unbounded. One can show that Δf is in the closure of the image of Δ_0 but it is not in $\operatorname{Im}\Delta_0$ (Exercise 7.2).

7.2.2 Weighted spaces on $\mathrm{Bl}_0\mathbf{C}^n$ and on $M \setminus \{p\}$

On $\mathrm{Bl}_0\mathbf{C}^n$ we can define weighted spaces identically to what we did on $\mathbf{R}^n$. Instead of the Laplacian, we will be interested in the operator

$$L_\eta : \varphi \mapsto -\Delta^2\varphi - R_\eta^{j\bar{k}}\partial_j\partial_{\bar{k}}\varphi,$$

where η is the Burns-Simanca metric. The mapping properties of this operator can be deduced from those of Δ^2 acting on $\mathbf{R}^{2n}$, using the fact that η is asymptotically flat. The result we will need is the following

Proposition 7.5. *If $\delta \in (4-2n, 0)$, then the operator*

$$L_\eta : C^{4,\alpha}_\delta(\mathrm{Bl}_0\mathbf{C}^n) \to C^{0,\alpha}_{\delta-4}(\mathrm{Bl}_0\mathbf{C}^n)$$

is an isomorphism.

Sketch of proof. For weights in the range $(4-2n, 0)$ the operator is self-adjoint, so we only need to show that it has trivial kernel. Suppose then that $L_\eta = -\mathcal{D}^*\mathcal{D}\varphi = 0$, and $\varphi \in C^{4,\alpha}_\delta$ for some $\delta < 0$. Since there are no indicial roots in $(4-2n, 0)$ it turns out that necessarily $\varphi \in C^{4,\alpha}_{4-2n}$. This decay is enough to show that the following integration by parts is possible:

$$0 = \int_{\mathrm{Bl}_0\mathbf{C}^n} \varphi\mathcal{D}^*\mathcal{D}\varphi\,\eta^n = \int_{\mathrm{Bl}_0\mathbf{C}^n} |\mathcal{D}\varphi|^2\,\eta^n,$$

so $\mathcal{D}\varphi = 0$. This implies that $\mathrm{grad}^{1,0}\varphi$ is a holomorphic vector field on $\mathrm{Bl}_0\mathbf{C}^n$, and it gives rise to a holomorphic vector field v on the complement $\mathbf{C}^n \setminus B$ of the unit ball B. The components of v are holomorphic functions, which by Hartog's theorem can be extended to all of $\mathbf{C}^n$. At the same time the components decay at infinity, so we must have $v = 0$. This implies that φ is constant, but φ also decays at infinity, so $\varphi = 0$. $\square$

We will also need analogous weighted spaces on $M_p := M \setminus \{p\}$. Let z^i be normal coordinates centered at p as before, defined for $|z| < 1$. We will write $B_r = \{z : |z| < r\}$ and fix a weight $\delta \in \mathbf{R}$. Given any $f : M_p \to \mathbf{R}$ and $r < 1/2$, define

$$f_r : B_2 \setminus B_1 \to \mathbf{R}$$
$$f_r(z) = r^{-\delta} f(rz),$$

as before. Using this we define the weighted Hölder norm

$$\|f\|_{C^{k,\alpha}_\delta(M_p)} = \|f\|_{C^{k,\alpha}(M_p\setminus B_{1/2})} + \sup_{r<1/2} \|f_r\|_{C^{k,\alpha}(B_2\setminus B_1)},$$

where on the annulus we can measure the Hölder norm using the Euclidean metric.

We are interested in the mapping properties of $\tilde{L}_\omega$ between these weighted spaces.

Proposition 7.6. *If $\delta \in (4-2n, 0)$, then*

$$\tilde{L}_\omega : C^{4,\alpha}_\delta(M_p) \to C^{0,\alpha}_{\delta-4}(M_p)$$

is an isomorphism.

Proof. Recall that we fixed a point q different from p, and

$$\tilde{L}_\omega(\varphi) = -\mathcal{D}^*\mathcal{D}\varphi - \varphi(q).$$

The operator $\mathcal{D}^*\mathcal{D}$ is self adjoint for weights in the range $(4-2n, 0)$, and by an almost identical argument to the proof of Proposition 7.5 we find that the kernel consists of the constant functions, under our assumption that M has no non-zero holomorphic vector fields. Therefore the cokernel of $\mathcal{D}^*\mathcal{D}$ consists of the constants, and from this one can check that $\tilde{L}_\omega$ is an isomorphism. □

7.2.3 Weighted spaces on $\mathrm{Bl}_p M$

The weighted spaces that we use on $\mathrm{Bl}_p M$ are essentially glued versions of the spaces defined on M_p and on $\mathrm{Bl}_0\mathbf{C}^n$. Recall that we have chosen normal coordinates z^i around p, defined for $|z| < 1$. To obtain $\mathrm{Bl}_p M$ we are gluing in a scaled down version of $\mathrm{Bl}_0\mathbf{C}^n$. In terms of the coordinates w^i on $\mathrm{Bl}_0\mathbf{C}^n$ we perform the gluing by identifying the annuli

$$\{r_\varepsilon < |z| < 2r_\varepsilon\} = \{\varepsilon^{-1}r_\varepsilon < |w| < 2\varepsilon^{-1}r_\varepsilon\}$$
$$z = \varepsilon w.$$

There are three regions of the manifold $\mathrm{Bl}_p M$:

- $M \setminus B_1$: Here the coordinates z^i are not defined, but we can think of it as the region where $|z| \geqslant 1$,
- $B_1 \setminus B_\varepsilon$: This region can either be thought of as a subset of M, where $\varepsilon \leqslant |z| < 1$, or also as a subset of $\mathrm{Bl}_0\mathbf{C}^n$, where $1 \leqslant |w| < \varepsilon^{-1}$.
- B_ε: Here the coordinates z^i are again not defined, and this region should be thought of as the subset of $\mathrm{Bl}_0\mathbf{C}^n$ where $|w| < 1$. Note that this region is not actually a ball, since it contains the exceptional divisor.

We define the weighted Hölder norms as follows. Suppose that $f : \mathrm{Bl}_p M \to \mathbf{R}$ and fix a weight δ. For $r \in (\varepsilon, 1/2)$ define

$$\begin{aligned} f_r &: B_2 \setminus B_1 \to \mathbf{R} \\ f_r(z) &= r^{-\delta} f(rz) \end{aligned}$$

and also let

$$\begin{aligned} f_\varepsilon &: \widetilde{B}_1 \subset \mathrm{Bl}_0 \mathbf{C}^n \to \mathbf{R} \\ f_\varepsilon(w) &= \varepsilon^{-\delta} f(\varepsilon w), \end{aligned}$$

where $\widetilde{B}_1$ is the subset of $\mathrm{Bl}_0\mathbf{C}^n$, where $|w| < 1$. We are abusing notation, writing $w \mapsto \varepsilon w$ for the map identifying $\widetilde{B}_1$ with $B_\varepsilon \subset \mathrm{Bl}_p M$. The weighted norm is defined as

$$\|f\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_p M)} = \|f\|_{C^{k,\alpha}(M\setminus B_1)} + \sup_{\varepsilon<r<1/2} \|f_r\|_{C^{k,\alpha}(B_2\setminus B_1)} + \|f_\varepsilon\|_{C^{k,\alpha}(\widetilde{B}_1)}.$$

On $M \setminus B_1$ and $\widetilde{B}_1$ we are measuring the Hölder norms with respect to fixed background metrics (or alternatively with respect to fixed coverings by charts). On $B_2 \setminus B_1$ we use the standard Euclidean metric.

Recall the cutoff functions γ_i that we used in Section 7.1.3. Since $\nabla\gamma_i$ is supported on $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$, and it is of order r_ε^{-1} on this annulus, one can check that

$$\|\gamma_i\|_{C^{4,\alpha}_0(\mathrm{Bl}_p M)} \leqslant c$$

for some constant c independent of ε. One use for these cutoff functions is that if f is a function on $\mathrm{Bl}_p M$, then $\gamma_1 f$ and $\gamma_2 f$ can be naturally thought of as functions on M_p and $\mathrm{Bl}_0\mathbf{C}^n$ respectively. Using these, an equivalent weighted norm could be defined as

$$\|f\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_p M)} = \|\gamma_1 f\|_{C^{k,\alpha}_\delta(M_p)} + \varepsilon^{-\delta}\|\gamma_2 f\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_0\mathbf{C}^n)}.$$

Note that the spaces $C^{k,\alpha}_\delta$ themselves do not depend on δ, as they all consist of functions on $\mathrm{Bl}_p M$ which are locally in $C^{k,\alpha}$. The weight δ only affects the norm. There are simple inequalities relating the norms for different weights:

$$\|f\|_{C^{k,\alpha}_{\delta'}} \leqslant \begin{cases} \|f\|_{C^{k,\alpha}_\delta} \text{ if } \delta' \leqslant \delta \\ \varepsilon^{\delta-\delta'}\|f\|_{C^{k,\alpha}_\delta} \text{ if } \delta' > \delta. \end{cases} \tag{70}$$

Lemma 7.7. *Let us write g_ε for the metric defined by ω_ε. We have*

$$\|g_\varepsilon\|_{C^{2,\alpha}_0(\mathrm{Bl}_p M)}, \|g_\varepsilon^{-1}\|_{C^{2,\alpha}_0(\mathrm{Bl}_p M)} \leqslant C,$$

for some C independent of ε.

Proof. We are measuring the Hölder norms of the components of g_ε and g_ε^{-1}. We can deal with the 3 regions of Bl_pM separately. For instance to deal with the annulus $B_{2r}\setminus B_r$, we need to pull back the components of g_ε to $B_2\setminus B_1$ (note that this is different from pulling back the metric itself, which would introduce a factor of r^2). The result follows since by the construction in Section 7.1.3 these pulled back metrics are uniformly equivalent to the Euclidean metric. $\square$

Lemma 7.8. *There are constants $c_0, C_1 > 0$ with the following property. If $\|\varphi\|_{C^{4,\alpha}_2} < c_0$, then $\omega_\varphi = \omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi$ is positive, and the corresponding metric g_φ satisfies*

$$\|g_\varphi - g_\varepsilon\|_{C^{2,\alpha}_{\delta-2}}, \|g_\varphi^{-1} - g_\varepsilon^{-1}\|_{C^{2,\alpha}_{\delta-2}}, \|\mathrm{Rm}_{g_\varphi} - \mathrm{Rm}_{g_\varepsilon}\|_{C^{0,\alpha}_{\delta-4}} < C_1\|\varphi\|_{C^{4,\alpha}_\delta}$$
$$\|L_{\omega_\varphi} - L_{\omega_\varepsilon}\|_{C^{4,\alpha}_\delta \to C^{0,\alpha}_{\delta-4}} < C_1\|\varphi\|_{C^{4,\alpha}_2},$$

where in the second line we are measuring the operator norm.

Proof. We have $\|\sqrt{-1}\partial\overline{\partial}\varphi\|_{C^{2,\alpha}_0} < C_2\|\varphi\|_{C^{4,\alpha}_2}$, so as long as c_0 is small enough, the form ω_φ is positive, and moreover we have

$$\|g_\varphi^{-1}\|_{C^{2,\alpha}_0} < 2C,$$

where C is as in the previous Lemma. The required estimates can then be obtained by straightforward calculations using multiplication properties of the weighted norms. For example

$$g_\varphi^{-1} - g_\varepsilon^{-1} = g_\varphi^{-1}(g_\varepsilon - g_\varphi)g_\varepsilon^{-1},$$

and so

$$\|g_\varphi^{-1} - g_\varepsilon^{-1}\|_{C^{2,\alpha}_{\delta_2}} \leqslant \|g_\varphi^{-1}\|_{C^{2,\alpha}_0}\|\sqrt{-1}\partial\overline{\partial}\varphi\|_{C^{2,\alpha}_{\delta-2}}\|g_\varepsilon^{-1}\|_{C^{2,\alpha}_0} < C_3\|\varphi\|_{C^{4,\alpha}_\delta}.$$

$\square$

The heart of the matter is the following result, which gives good bounds on the inverse of the linearization of our equation. The idea is that we can glue the inverses of the linear operators from Propositions 7.5 and 7.6.

Theorem 7.9. *Choose $\delta \in (4-2n, 0)$. There exists a $C > 0$ independent of ε, such that for sufficiently small $\varepsilon > 0$ the operator*

$$\begin{aligned}\tilde{L}_{\omega_\varepsilon} : C^{4,\alpha}_\delta(\mathrm{Bl}_pM) &\to C^{0,\alpha}_{\delta-4}(\mathrm{Bl}_pM)\\ \varphi &\mapsto -\Delta^2_{\omega_\varepsilon}\varphi - R^{j\bar{k}}_{\omega_\varepsilon}\partial_j\partial_{\bar{k}}\varphi - \varphi(q)\end{aligned}$$

is invertible, and we have a bound $\|\tilde{L}^{-1}_{\omega_\varepsilon}\| < C$ for its inverse.

Proof. We first construct an approximate inverse for $\tilde{L}_{\omega_\varepsilon}$. In addition to the cutoff functions γ_i we will need two more cutoff functions β_i. Let us write $\alpha = \frac{n-1}{n}$, so $r_\varepsilon = \varepsilon^\alpha$. Choose $\underline{\alpha}$ and $\overline{\alpha}$ satisfying

$$0 < \underline{\alpha} < \alpha < \overline{\alpha} < 1.$$

We define

$$\beta_1(z) = \beta\left(\frac{\log|z|}{\log\varepsilon}\right),$$

where $\beta : \mathbf{R} \to \mathbf{R}$ is a fixed cutoff function such that $\beta(r) = 1$ for $r < a$ and $\beta(r) = 0$ for $r > \overline{a}$. Then β_1 is supported on $M \setminus B_{\varepsilon^{\overline{\alpha}}}$, and it equals 1 on the support of γ_1. An important point is that

$$\|\nabla\beta_1\|_{C^{3,\alpha}_{-1}} \leqslant \frac{C}{|\log\varepsilon|}. \tag{71}$$

The function β_2 is defined similarly, so that β_2 is supported in $B_{\varepsilon^{\underline{\alpha}}}$, and is equal to 1 on the support of γ_2. The function β_2 will also satisfy the estimate (71).

The approximate inverse is the bounded operator

$$P : C^{0,\alpha}_{\delta-4}(\mathrm{Bl}_p M) \to C^{4,\alpha}_{\delta}(\mathrm{Bl}_p M)$$

defined as follows. For $\varphi \in C^{0,\alpha}_{\delta-4}$ we let

$$P(\varphi) = \beta_1\tilde{L}^{-1}_\omega(\gamma_1\varphi) + \varepsilon^{-4}\beta_2 L^{-1}_\eta(\gamma_2\varphi),$$

using the inverse linear operators given by Propositions 7.5 and 7.6. Here we use the fact that $\gamma_1\varphi$ and $\gamma_2\varphi$ can be thought of as functions on M_p and $\mathrm{Bl}_0\mathbf{C}^n$ respectively, and conversely if we have functions on these spaces, we can view them as functions on $\mathrm{Bl}_p M$ once we multiply them by β_1, β_2.

This opeartor P gives an approximate inverse to $\tilde{L}_{\omega_\varepsilon}$ in the sense that

$$\|\tilde{L}_{\omega_\varepsilon}P(\varphi) - \varphi\|_{C^{0,\alpha}_{\delta-4}} \leqslant o(1)\|\varphi\|_{C^{0,\alpha}_{\delta-4}}, \tag{72}$$

where $o(1)$ denotes a constant which goes to zero as $\varepsilon \to 0$. Assuming (72), we can choose ε sufficiently small, so that

$$\|\tilde{L}_{\omega_\varepsilon} \circ P - \mathrm{Id}\| < \frac{1}{2}$$

in terms of the operator norm for maps $C^{0,\alpha}_{\delta-4} \to C^{0,\alpha}_{\delta-4}$. It follows that $\tilde{L}_{\omega_\varepsilon} \circ P$ is invertible, with inverse bounded by 2. Then $\tilde{L}_{\omega_\varepsilon}$ is also invertible (it has a right-inverse, but also it has index zero), and

$$\tilde{L}^{-1}_{\omega_\varepsilon} = P \circ (\tilde{L}_{\omega_\varepsilon} \circ P)^{-1}$$

is bounded by a constant C independent of ε.

What remains is to verify (72). We have

$$\tilde{L}_{\omega_\varepsilon} P(\varphi) - \varphi = \tilde{L}_{\omega_\varepsilon}(\beta_1 \tilde{L}_\omega^{-1}(\gamma_1\varphi)) - \gamma_1\varphi + \varepsilon^{-4}\tilde{L}_{\omega_\varepsilon}(\beta_2 \tilde{L}_\eta^{-1}(\gamma_2\varphi)) - \gamma_2\varphi. \quad (73)$$

Since

$$\tilde{L}_\omega(\beta_1 \tilde{L}_\omega^{-1}(\gamma_1\varphi)) = \beta_1\gamma_1\varphi + \text{ terms involving } \nabla\beta_1 \ ,$$

the bound (71) together with $\beta_1\gamma_1 = \gamma_1$ can be used to get

$$\|\tilde{L}_\omega(\beta_1 \tilde{L}_\omega^{-1}(\gamma_1\varphi)) - \gamma_1\varphi\|_{C^{0,\alpha}_{\delta-4}} \leqslant o(1)\|\varphi\|_{C^{0,\alpha}_{\delta-4}}.$$

In addition Lemma 7.8 can be used to see that on the support of β_1 we have $\|L_{\omega_\varepsilon} - L_\omega\| \leqslant o(1)$, so

$$\|\tilde{L}_{\omega_\varepsilon}(\beta_1 \tilde{L}_\omega^{-1}(\gamma_1\varphi)) - \gamma_1\varphi\|_{C^{0,\alpha}_{\delta-4}} \leqslant o(1)\|\varphi\|_{C^{0,\alpha}_{\delta-4}}.$$

A similar argument can be used to deal with the other terms in (73), noting that $L_{\varepsilon^2\eta} = \varepsilon^{-4}L_\eta$, and that on the support of β_2 the metric ω_ε is approximately equal to $\varepsilon^2\eta$. $\square$

7.3 Solving the non-linear equation

We are now ready to solve the equation

$$S(\omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi) - S(\omega) - \varphi(q) = 0, \quad (74)$$

for sufficiently small ε where q is a point outside the unit ball around p. We fix $\delta \in (4-2n, 0)$, and eventually we will choose δ to be very close to 0. Writing

$$S(\omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi) = S(\omega_\varepsilon) + L_{\omega_\varepsilon}(\varphi) + Q_{\omega_\varepsilon}(\varphi), \quad (75)$$

Equation (74) is equivalent to

$$S(\omega_\varepsilon) - S(\omega) + L_{\omega_\varepsilon}(\varphi) - \varphi(q) + Q_{\omega_\varepsilon}(\varphi) = 0.$$

From Theorem 7.9 we know that the operator

$$\tilde{L}_{\omega_\varepsilon}(\varphi) := L_{\omega_\varepsilon}(\varphi) - \varphi(q)$$

has a bounded inverse, in terms of which our equation can be rewritten as

$$\varphi = \tilde{L}_{\omega_\varepsilon}^{-1}\Big(S(\omega) - S(\omega_\varepsilon) - Q_{\omega_\varepsilon}(\varphi)\Big).$$

Let us define the operator $\mathcal{N}$ by

$$\begin{aligned}\mathcal{N} : C^{4,\alpha}_{\delta}(\mathrm{Bl}_p M) &\to C^{4,\alpha}_{\delta}(\mathrm{Bl}_p M)\\ \varphi &\mapsto \tilde{L}^{-1}_{\omega_\varepsilon}\Big(S(\omega) - S(\omega_\varepsilon) - Q_{\omega_\varepsilon}(\varphi)\Big).\end{aligned}$$

Equation (74) is then equivalent to the fixed point problem $\varphi = \mathcal{N}(\varphi)$.

The following Lemma shows that $\mathcal{N}$ is a contraction on a suitable set.

Lemma 7.10. *There is a constant $c_1 > 0$ such that if $\|\varphi\|_{C^{4,\alpha}_2}, \|\psi\|_{C^{4,\alpha}_2} \leqslant c_1$, then*

$$\|\mathcal{N}(\varphi) - \mathcal{N}(\psi)\|_{C^{4,\alpha}_\delta} \leqslant \frac{1}{2}\|\varphi - \psi\|_{C^{4,\alpha}_\delta}.$$

Proof. We have

$$\mathcal{N}(\varphi) - \mathcal{N}(\psi) = \tilde{L}^{-1}_{\omega_\varepsilon}(Q_{\omega_\varepsilon}(\psi) - Q_{\omega_\varepsilon}(\varphi)).$$

By the mean value theorem there is a $t \in [0,1]$ such that $\chi = t\varphi + (1-t)\psi$ satisfies

$$Q_{\omega_\varepsilon}(\psi) - Q_{\omega_\varepsilon}(\varphi) = DQ_{\omega_\varepsilon,\chi}(\psi - \varphi).$$

Differentiating (75) at χ we have

$$DQ_{\omega_\varepsilon,\chi} = L_{\omega_\chi} - L_{\omega_\varepsilon},$$

so from Lemma 7.8 we know that if $\|\chi\|_{C^{4,\alpha}_2} < c_0$, then

$$\begin{aligned}\|Q_{\omega_\varepsilon}(\psi) - Q_{\omega_\varepsilon}(\varphi)\|_{C^{0,\alpha}_{\delta-4}} &< C\|\chi\|_{C^{4,\alpha}_2}\|\psi - \varphi\|_{C^{4,\alpha}_\delta}\\ &\leq C\{\|\varphi\|_{C^{4,\alpha}_2} + \|\psi\|_{C^{4,\alpha}_2}\}\|\psi - \varphi\|_{C^{4,\alpha}_\delta}.\end{aligned}$$

Since $\tilde{L}^{-1}_{\omega_\varepsilon}$ is bounded independently of ε, the result follows once c_1 is chosen small enough. $\square$

We also need to know how good our approximate solution ω_ε is.

Lemma 7.11. *For sufficiently small ε we have*

$$\|S(\omega_\varepsilon) - S(\omega)\|_{C^{0,\alpha}_{\delta-4}} \leqslant Cr_\varepsilon^{4-\delta}$$

for some constant C.

Proof. We examine 3 different regions of $\mathrm{Bl}_p M$. On $M \setminus B_{2r_\varepsilon}$ the metrics ω_ε and ω are equal, so $S(\omega_\varepsilon) - S(\omega) = 0$.

On $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$, in terms of the Euclidean metric ω_E we have

$$\omega_E = \omega_\varepsilon + \sqrt{-1}\partial\overline{\partial}\varphi,$$

where $\varphi = O(|z|^4)$. It follows that

$$\|\varphi\|_{C^{4,\alpha}_\delta(B_{2r_\varepsilon}\setminus B_{r_\varepsilon})} \leqslant C_1 r_\varepsilon^{4-\delta}$$

for some constant C_1. Using this for $\delta = 2$ as well, Lemma 7.8 implies that

$$\|S(\omega_\varepsilon) - 0\|_{C^{0,\alpha}_{\delta-4}(B_{2r_\varepsilon}\setminus B_{r_\varepsilon})} \leqslant C_2 r_\varepsilon^{4-\delta},$$

for sufficiently small ε. Since $S(\omega)$ is a fixed constant, we also have

$$\|S(\omega)\|_{C^{0,\alpha}_{\delta-4}(B_{2r_\varepsilon}\setminus B_{r_\varepsilon})} \leqslant C_3 r_\varepsilon^{4-\delta},$$

so this takes care of the annulus $B_{2r_\varepsilon} \setminus B_{r_\varepsilon}$.

On B_{r_ε} we have $S(\omega_\varepsilon) = 0$, and again

$$\|S(\omega)\|_{C^{0,\alpha}_{\delta-4}(B_{r_\varepsilon})} \leqslant C_3 r_\varepsilon^{4-\delta}.$$

□

We can finally put the pieces together.

Proposition 7.12. *Using the constant c_1 from Lemma 7.10, let*

$$\mathcal{U} = \left\{\varphi \in C^{4,\alpha}_\delta \,:\, \|\varphi\|_{C^{4,\alpha}_\delta} \leqslant c_1\varepsilon^{2-\delta}\right\} \subset C^{4,\alpha}_\delta(\mathrm{Bl}_p M).$$

If ε is sufficiently small, then $\mathcal{N}$ is a contraction on $\mathcal{U}$, and $\mathcal{N}(\mathcal{U}) \subset \mathcal{U}$. In particular $\mathcal{N}$ has a fixed point, which gives a cscK metric on $\mathrm{Bl}_p M$ in the Kähler class $\pi^*[\omega] - \varepsilon^2[E]$.

Proof. From the comparison (70) between the weighted norms we have $\|\varphi\|_{C^{4,\alpha}_2} \leqslant c_1$ if $\varphi \in \mathcal{U}$. From Lemma 7.10 it follows then that $\mathcal{N}$ is a contraction on $\mathcal{U}$, and in addition

$$\begin{aligned}\|\mathcal{N}(\varphi)\|_{C^{4,\alpha}_\delta} &\leqslant \|\mathcal{N}(\varphi) - \mathcal{N}(0)\|_{C^{4,\alpha}_\delta} + \|\mathcal{N}(0)\|_{C^{4,\alpha}_\delta} \\ &\leqslant \frac{1}{2}\|\varphi\|_{C^{4,\alpha}_\delta} + \|\mathcal{N}(0)\|_{C^{4,\alpha}_\delta}.\end{aligned} \tag{76}$$

From Lemma 7.11 we have

$$\|\mathcal{N}(0)\|_{C^{4,\alpha}_\delta} \leqslant C\|S(\omega_\varepsilon) - S(\omega)\|_{C^{0,\alpha}_{\delta-4}} \leqslant C' r_\varepsilon^{4-\delta}.$$

From the definition of r_ε,

$$r_\varepsilon^{4-\delta} = \varepsilon^{(4-\delta)\frac{n-1}{n}},$$

and if δ is close to 0 and $n > 2$, then

$$(4-\delta)\frac{n-1}{n} > 2 - \delta.$$

It follows that if ε is sufficiently small, we have

$$\|\mathcal{N}(0)\|_{C^{4,\alpha}_\delta} \leqslant \frac{1}{2} c_1 \varepsilon^{2-\delta}.$$

From (76) we then have $\mathcal{N}(\varphi) \in \mathcal{U}$. This completes the proof. □

7.4 Remarks

Theorem 7.1 gives a way of constructing new cscK manifolds. For instance we could take M to be a Kähler-Einstein manifold of dimension greater than 2, given by Theorem 3.1. It is not hard to see that such M does not admit holomorphic vector fields. We can then obtain new cscK metrics on the blowup of M at any point, and we can even iterate the construction. Note that we have little understanding of what the metrics produced by Theorem 3.1 actually look like. In contrast the perturbation method giving Theorem 7.1 implies that the metrics we obtain on the blowup $\mathrm{Bl}_p M$ are very close to our original metric on M away from the point p, while near p they are very close to scaled down versions of the Burns-Simanca metric.

The condition $n > 2$ in Theorem 7.1 can be relaxed to $n \geqslant 2$ without too much difficulty. Since $4 - 2n = 0$ for $n = 2$ we can not work with weights in the interval $(4-2n, 0)$, but instead one can either work with $\delta \in (-1, 0)$ or $\delta \in (0, 1)$. Choosing $\delta < 0$ introduces some extra complications when dealing with the linear operator on $\mathrm{Bl}_0\mathbf{C}^2$, while choosing $\delta > 0$ causes complications on M_p.

A more serious restriction is the assumption that there are no non-zero holomorphic vector fields. If we allow holomorphic vector fields, then there are obstructions for the blowup $\mathrm{Bl}_p M$ admitting cscK metrics, or even extremal metrics. Various results in this direction have been obtained in [2, 3, 4, 46]. The expectation is that K-stability of the blowup is the only obstruction, but the current results do not yet get this far except in the case when we are blowing up a Kähler-Einstein manifold [46].

In addition to giving new examples of cscK manifolds, Theorem 7.1 also has theoretical applications. One important application is the following sharpening of Corollary 6.17 due to Stoppa [45].

Theorem 7.13. *Suppose that the compact Kähler manifold M has no holomorphic vector fields, and $\omega \in c_1(L)$ is a cscK metric. Then (M, L) is K-stable.*

Sketch of proof. Recall that a test-configuration χ for M is an embedding $M \subset \mathbf{CP}^N$ using a basis of sections of L^r for some r, together with a $\mathbf{C}^*$-action on $\mathbf{CP}^N$. We have defined the Donaldson-Futaki invariant $DF(\chi)$, and the norm $\|\chi\|$, and we need to show that $DF(\chi) > 0$ whenever $\|\chi\| > 0$.

From Corollary 6.17 we already know that $DF(\chi) \geqslant 0$, so let us suppose by contradiction that $DF(\chi) = 0$ and $\|\chi\| > 0$. For any point $p \in M$ and large k, the line bundle $kL - E$ is an ample line bundle on the blowup $\mathrm{Bl}_p M$, and the test-configuration χ induces a test-configuration $\hat{\chi}$ for the pair $(\mathrm{Bl}_p M, kL - E)$. The strategy of [45] is to choose the $p \in M$ in such a way, that for sufficiently large k we have $DF(\hat{\chi}) < 0$. We know from Theorem 7.1 that $\mathrm{Bl}_p M$ admits a cscK metric in $c_1(kL - E)$ for sufficiently large k, so this contradicts Corollary 6.17. $\square$

7.5 Exercises

Exercise 7.1. Suppose that $u : \mathbf{R}^n \to \mathbf{R}$ is in the weighted space $C^{k-2,\alpha}_{\delta-2}$ for some $\delta \in (2-n, 0)$, $k \geqslant 2$, $\alpha \in (0,1)$ and $n > 2$. Show that the function $f : \mathbf{R}^n \to \mathbf{R}$ defined by

$$f(x) = \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-2}} u(y)\, dy$$

is well-defined, and

$$\|f\|_{C^{k,\alpha}_\delta} \leqslant C \|u\|_{C^{k-2,\alpha}_{\delta-2}}$$

for some C independent of u.

Exercise 7.2. Consider the Laplacian acting between weighted spaces on $\mathbf{R}^n$:

$$\Delta_\delta : C^{k,\alpha}_\delta(\mathbf{R}^n) \to C^{k-2,\alpha}_{\delta-2}(\mathbf{R}^n).$$

(a) Show that if $\delta = 0$, then the image of Δ_δ is not closed.

(b) Using exercise 2.5 show that for $\delta \in (2-n, 0)$ the image of Δ_δ is closed.

References

[1] V. Apostolov, D. M. J. Calderbank, P. Gauduchon, and C. W. Tønnesen-Friedman. Hamiltonian 2-forms in Kähler geometry III, extremal metrics and stability. *Invent. Math.*, 173(3):547–601, 2008.

[2] C. Arezzo and F. Pacard. Blowing up and desingularizing constant scalar curvature Kähler manifolds. *Acta Math.*, 196(2):179–228, 2006.

[3] C. Arezzo and F. Pacard. Blowing up Kähler manifolds with constant scalar curvature II. *Ann. of Math. (2)*, 170(2):685–738, 2009.

[4] C. Arezzo, F. Pacard, and M. A. Singer. Extremal metrics on blow ups. *Duke Math. J.*, 157(1):1–51, 2011.

[5] T. Aubin. Métriques riemanniennes et courbure. *J. Differential Geom.*, 4:383–424, 1970.

[6] T. Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[7] Z. Błocki. The Calabi-Yau theorem. *http://gamma.im.uj.edu.pl/~blocki/publ/ln/tln.pdf*.

[8] E. Calabi. Extremal Kähler metrics. In S. T. Yau, editor, *Seminar on Differential Geometry*. Princeton, 1982.

[9] D. Catlin. The Bergman kernel and a theorem of Tian. In *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., pages 1–23. Birkhäuser Boston Inc., 1999.

[10] X. X. Chen. Space of Kähler metrics III. On the lower bound of the Calabi energy and geodesic distance. *Invent. Math.*, 175(3):453–503, 2009.

[11] X. X. Chen, C. LeBrun, and B. Weber. On conformally Kähler, Einstein manifolds. *J. Amer. Math. Soc.*, 21(4):1137–1168, 2008.

[12] X. X. Chen and G. Tian. Geometry of Kähler metrics and foliations by holomorphic discs. *Publ. Math. Inst. Hautes Études Sci.*, (107):1–107, 2008.

[13] S. K. Donaldson. Lecture notes for TCC course "Geometric Analysis". *http://www2.imperial.ac.uk/~skdona/GEOMETRICANALYSIS.PDF*.

[14] S. K. Donaldson. Remarks on gauge theory, complex geometry and four-manifold topology. In Atiyah and Iagolnitzer, editors, *Fields Medallists' Lectures*, pages 384–403. World Scientific, 1997.

[15] S. K. Donaldson. Symmetric spaces, Kähler geometry and Hamiltonian dynamics. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–33. Amer. Math. Soc., Providence, RI, 1999.

[16] S. K. Donaldson. Scalar curvature and stability of toric varieties. *J. Differential Geom.*, 62:289–349, 2002.

[17] S. K. Donaldson. Lower bounds on the Calabi functional. *J. Differential Geom.*, 70(3):453–472, 2005.

[18] S. K. Donaldson. Discussion of the Kähler-Einstein problem. *http://www2.imperial.ac.uk/~skdona/KENOTES.PDF*, 2009.

[19] A. Fujiki. Moduli space of polarized algebraic manifolds and Kähler metrics [translation of Sûgaku **42** (1990), no. 3, 231–243;]. *Sugaku Expositions*, 5(2):173–191, 1992.

[20] A. Futaki. An obstruction to the existence of Einstein-Kähler metrics. *Invent. Math.*, 73:437–443, 1983.

[21] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, reprint of the 1998 edition, 2001.

[22] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[23] A. Hwang and M. A. Singer. A momentum construction for circle-invariant Kähler metrics. *Trans. Amer. Math. Soc.*, 354(6):2285–2325, 2002.

[24] C. LeBrun. Counter-examples to the generalized positive action conjecture. *Comm. Math. Phys.*, 118(4):591–596, 1988.

[25] C. LeBrun and S. R. Simanca. On the Kähler classes of extremal metrics. In *Geometry and global analysis (Sendai, 1993)*, pages 255–271. Tohoku Univ., 1993.

[26] C. LeBrun and S. R. Simanca. Extremal Kähler metrics and complex deformation theory. *Geom. and Func. Anal.*, 4(3):298–336, 1994.

[27] L. Lempert and L. Vivas. Geodesics in the space of Kähler metrics. *arXiv:1105.2188.*

[28] C. Li and C. Xu. Special test configurations and K-stability of Fano varieties. *in preparation.*

[29] R. B. Lockhart and R. C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)*, 1(3):409–447, 1985.

[30] Z. Lu. On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch. *Amer. J. Math.*, 122(2):235–273, 1998.

[31] T. Mabuchi. K-stability of constant scalar curvature polarization. arXiv:0812.4093.

[32] T. Mabuchi. K-energy maps integrating Futaki invariants. *Tohoku Math. J.*, 38(4):575–593, 1986.

[33] T. Mabuchi. Some symplectic geometry on compact Kähler manifolds. *Osaka J. Math.*, 24:227–252, 1987.

[34] Y. Matsushima. Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne. *Nagoya Math. J.*, 11:145–150, 1957.

[35] D. McDuff and D. Salamon. *Introduction to symplectic topology.* OUP, 1998.

[36] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[37] F. Pacard. Lectures on connected sum constructions in geometry and nonlinear analysis. *http://www.math.polytechnique.fr/~pacard/Publications/Lecture-Part-I.pdf*.

[38] S. T. Paul and G. Tian. Algebraic and analytic K-stability. *arXiv:math/0405530v1.*

[39] D. H. Phong, N. Sesum, and J. Sturm. Multiplier ideal sheaves and the Kähler-Ricci flow. *Comm. Anal. Geom.*, 15(3):613–632, 2007.

[40] W.-D. Ruan. Canonical coordinates and Bergman metrics. *Comm. Anal. Geom.*, 6(3):589–631, 1998.

[41] S. Semmes. Complex Monge-Ampère equations and symplectic manifolds. *Amer. J. Math.*, 114:495–550, 1992.

[42] S. R. Simanca. Kähler metrics of constant scalar curvature on bundles over $C\mathrm{P}_{n-1}$. *Math. Ann.*, 291(2):239–246, 1991.

[43] L. Simon. *Theorems on regularity and singularity of energy minimizing maps*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996.

[44] Y. T. Siu. *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*. DMV Seminar 8. Birkhäuser Verlag, Basel, 1987.

[45] J. Stoppa. K-stability of constant scalar curvature Kähler manifolds. *Adv. Math.*, 221(4):1397–1408, 2009.

[46] G. Székelyhidi. On blowing up extremal Kähler manifolds. *to appear in Duke Math. J.*

[47] G. Székelyhidi. Extremal metrics and K-stability. *Bull. Lond. Math. Soc.*, 39(1):76–84, 2007.

[48] R. P. Thomas. Notes on GIT and symplectic reduction for bundles andvarieties. Surveys in Differential Geometry, 10 (2006): A Tribute to Professor S.-S. Chern.

[49] G. Tian. On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.*, 32(1):99–130, 1990.

[50] G. Tian. On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, 101(1):101–172, 1990.

[51] G. Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 137:1–37, 1997.

[52] G. Tian. *Canonical metrics in Kähler geometry.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2000.

[53] C. W. Tønnesen-Friedman. Extremal Kähler metrics on minimal ruled surfaces. *J. Reine Angew. Math.*, 502:175–197, 1998.

[54] X. Wang. Moment map, Futaki invariant and stability of projective manifolds. *Comm. Anal. Geom.*, 12(5):1009–1037, 2004.

[55] S.-T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I. *Comm. Pure Appl. Math.*, 31:339–411, 1978.

[56] S.-T. Yau. Open problems in geometry. *Proc. Symposia Pure Math.*, 54:1–28, 1993.

[57] S. Zelditch. Szegő kernel and a theorem of Tian. *Int. Math. Res. Notices*, 6:317–331, 1998.