Rational Homotopy Theory Seminar

Week 1: Minicourse Part I

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The minicourse will be based on the notes from a course on rational homotopy theory taught at the University of Copenhagen by Alexander Berglund. Additional resources are Sullivan's "Infinitesimal Computations..." and Quillen's "Rational Homotopy Theory".

Outline of minicourse:

1. (Chapters 1,2,4,5,9)
   - Basic rational homotopy theory
   - Rational homotopy groups of spheres
   - General picture and main results
2. Simplicial objects and differential graded algebras (Chapters 6,7)
3. (Chapter 8)
   - Sullivan's approach to rational homotopy theory
   - Sullivan models
   - Commutative differential graded algebras
4. (Chapter 10)
   - Quillen's approach
   - Differential graded Lie algebras

From now on, let X be a simply connected CW complex of finite type.

Theorem. (Serre) The homotopy groups $\pi_k(X)$ are finitely generated abelian groups for all $k \geq 0$.

Therefore we can express the homotopy groups of X as

$$\pi_n(X) = \mathbb{Z}^r \oplus T$$

and we have

$$\pi_n(X) \otimes \mathbb{Q} \cong \mathbb{Q}^r.$$

We want a functor $X \mapsto X_{\mathbb{Q}}$ such that $\pi_n(X_{\mathbb{Q}}) \cong \pi_n(X) \otimes \mathbb{Q}$. This will lead us to define *rational equivalence* of spaces as

$$X \sim_{\mathbb{Q}} Y \iff X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}.$$

Recall that if A is an abelian group, then there exists a unique (up to homotopy) connected CW complex $K(A, n)$ such that there is a natural bijection

$$[X, K(A, n)] \cong H^n(X; A).$$

Equivalently, the space $K(A, n)$ is characterized by the property that $\pi_* K(A, n) \cong A$ if $* = n$ and $\pi_* K(A, n) \cong 0$ if $* \neq n$.

Note that we can't build any space as a product of Eilenberg-Mac Lane spaces, i.e. in general we don't have $X = \prod_n K(A, n)$ since having the same homotopy groups does not imply having the same homotopy type. However, one can build any space X using a "twisted product", i.e. a *Postnikov tower.*

[diagram of a Postnikov tower starting at $X \to X_2$]

In the Postnikov tower, one requires that $\pi_k \xrightarrow{cong} \pi_k(X_n)$ for $k \leq n$ and $\pi_k(X_n) = 0$ for $k > n$.

If X is CW, then

[diagram]

is a pullback.

Rationalization

Theorem. Let $X \to Y$ be a map between simply connected spaces. Then the following are equivalent:

1. $f_n : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ is an isomorphism for all n
2. $f_n : H_n(X; \mathbb{Q}) \to H_n(Y; \mathbb{Q})$ is an isomorphism for all n
3. $f_n : H_n(X) \otimes \mathbb{Q} \to H_n(Y) \otimes \mathbb{Q}$ is an isomorphism for all n

Proof. Long exact sequences in homotopy and homology plus the Serre spectral sequence. □

A map $f : X \to Y$ is a *rational homotopy equivalence* if any of the above holds. We say that X and Y are *rational homotopy equivalent* if there exists a zigzag of rational homotopy equivalences between them.

Recall that an abelian group A is *uniquely divisible* if the map $A \xrightarrow{\cdot n} A$ is a surjection for all $n \in \mathbb{Z} - 0$. For example, $\mathbb{Q}$ is uniquely divisible. Equivalently, the map $A \to A \otimes \mathbb{Q}$ is an isomorphism.

Theorem. The following are equivalent:

1. $\pi_k(X)$ is uniquely divisible for all k
2. $H_k(X)$ is uniquely divisible for all k

We say that X is *rational* if the above hold. Equivalently, if X is $\mathbb{Q}$-local. A *rationalization* of X is a map $X \xrightarrow{r} X_{\mathbb{Q}}$ where $X_{\mathbb{Q}}$ is rational and r is a rational homotopy equivalence. We say that $X_{\mathbb{Q}}$ is a *rationalization* of X.

Theorem. Every simply connected space has a rationalization.
Proof. Induction up the Postnikov tower for X. □

Theorem. The following are equivalent:

1. X is rational

2. For all rational homotopy equivalences $r : Z \to Y$ between CW spaces, there is a bijection
$$r^* : [Y, X] \xrightarrow{\cong} [Z, X].$$

[chart of unstable homotopy groups of spheres]

Theorem. (Serre) $\pi_k(S^n) = \mathbb{Z} \oplus T$ if $k = n$ with n odd or $k = n$, $2n - 1$ with n even. Outside of these indices, $\pi_k(S^n) = T$.

Recall that $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[x]$ with $|x| = n$ if n is even, and $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \Lambda(x)$ if n is odd. In other words, $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ is a free graded commutative algebra on one generator in degree n.

Theorem, restated. $\pi_k(S^n_{\mathbb{Q}})$ is a free graded Whitehead algebra with one generator $\iota \in \pi_n(S^n_{\mathbb{Q}})$ which is the class of the identity.

A *Whitehead algebra* is a graded $\mathbb{Q}$-vector space π_* together with a bilinear map $[-,-] : \pi_p \otimes \pi_q \to \pi_{p+q-1}$ satisfying

1. (Symmetry) $[X, Y] = (-1)^{pq}[Y, X]$

2. (Jacobi) $(-1)^{pr}[[\alpha, \beta], \gamma] + (-1)^{qp}[[\beta, \gamma], \alpha] + (-1)^{rq}[[\gamma, \alpha], \beta] = 0$.

Compare this definition with Lie algebras or the Samuelson product.

Next time: We begin talking about the chain of Quillen equivalences (where each category must have some restrictions)

$$Top_{\mathbb{Q}} \leftrightarrows sSet_{\mathbb{Q}} \leftrightarrows CDGA_{\mathbb{Q}} \leftrightarrows CDGC_{\mathbb{Q}} \leftrightarrows DGL_{\mathbb{Q}}$$