Rational Homotopy Theory Seminar

Week 10: Formality of Kähler Manifolds

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Crash course on Kähler manifolds. Let M be an n-dimensional manifold. A *Kähler structure* on M consists of the following data:

1. An almost-complex structure $J : TM \to TM$ such that $J^2 = -1$. Note that we can write
$$M \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$$
where J acts on the first component (the *holomorphic part*) by i and the second component (the *antiholomorphic part*) by $-i$. Furthermore, we have
$$\overline{T^{1,0}M} \cong T^{0,1}M$$
which induces a splitting at the level of forms
$$\Omega^\bullet(M) \cong \bigoplus_{p+q=\bullet} \Omega^{p,q}M$$
and a splitting of the differential
$$d = \partial + \overline{\partial}.$$

2. A Riemannian metric g which gives rise to a 2-form
$$\omega(-,-) = g(-, J(-))$$

This data must satisfy the following conditions:

1. $\overline{\partial}^2 = 0$ ([NN] says that then (M, J) is a $\mathbb{C}$-manifold)

2. $g(X, Y) = g(JX, JY)$

3. One of the following equivalent conditions holds:

   (a) (topological) $d\omega = 0$. Equivalently, (M, ω) is symplectic

   (b) (geometric) $\nabla^g J = 0$ where ∇^g is a Levi-Civita connection. More explicitly, this means that parallel translation preserves the almost-complex structure.

   (c) (analytic) Locally, $\omega = i\overline{\partial}\partial f$ where $f : M \to \mathbb{R}$ is a smooth function. For example, if one takes $\mathbb{D}^n \subset \mathbb{C}^n$, then $\omega = \frac{1}{2\pi}\overline{\partial}\partial(1 - |z|^2)$

   (d) (analytic/geometric) Locally, $g_m = g_{\mathbb{C}^n} + O(|z|^2)$. In other words, "any identity involving g and its derivatives is valid on M if and only if it is valid on flat $\mathbb{C}^n$.

   (e) The holonomy is contained in $SU(n)$

Properties. From this information, you can define

$$d^c = \sqrt{-1}(\overline{\partial} - \partial) = J^{-1}dJ$$

which satisfies

- $(d^c)^2 = 0$
- $[d, d^c] = 0$
- $-dd^c = 2i\partial\overline{\partial}$

Example. Given local coordinates on $\mathbb{C}$, $z = re^{i\theta}$, we have

$$d^c = rd\theta \otimes \partial_r + \frac{1}{4}dr \otimes \partial_\theta$$

and

$$dd^c = dx \wedge dy \otimes (\partial_x^2 + \partial_y^2)$$

Recall. If D is a differential operator, we can define its *Laplacian* as

$$\Delta_D = [D, D^*].$$

Moreover, the *deRham Laplacian* for complex manifolds satisfies

$$\Delta_d = \Delta_{\overline{\partial}} + \Delta_\partial$$

and if M is Kähler, we have

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_\partial = \Delta_{d^c}.$$

Example. Using (d) above and a computation on $\mathbb{C}^n$, we have

$$[\overline{\partial}, \partial^*] = [partial, \overline{\partial}^*] = 0.$$

Main Lemma. (the dd^c-lemma) Suppose $\alpha \in \Omega(M)$ is a form on M such that $d\alpha = d^c\alpha = 0$ and $\alpha = d\gamma$ or $\alpha = d^c\gamma$, then there exists some β such that

$$\alpha = dd^c\beta.$$

In other words, we have

$$ker(d) \cap ker(d^c) \cap im(d) = im(dd^c).$$

Proof. Define the projection to the zero-th eigenspace $\mathcal{H} : \Omega^\bullet(M) \to Ker(\Delta_D)$ where $D = \partial, \overline{\partial}, d, d^c$. Standard elliptic PDE theory on manifolds implies that we have a decomposition

$$\Omega^\bullet(M) = \bigoplus_{\lambda \in Spec(\Delta_D)} V_\lambda$$

into eigenspaces of the the Laplacian Δ_D. Notice that on $(ker\Delta_D)^\perp$ one can invert Δ_D to obtain the Green's function for D satisfying

$$G_D = 0 \quad \text{on } V_0$$

$$G_D = \frac{1}{\lambda} \quad \text{on } V_\lambda.$$

In particular $[G_D, D^*] = 0$.

Hodge decomposition then tells that for any form ψ, we can write

$$\psi = \mathcal{H}\psi + \Lambda_D G_D \psi.$$

By the assumptions of the lemma, we have

$$\mathcal{H}_d(\alpha) = \mathcal{H}_{d^c}(\alpha) = 0$$

so therefore

$$\alpha = dd^* G_d \alpha$$

and

$$\alpha = d^c (d^c)^* G_{d^c} \alpha.$$

Calculation then gives an "explicit" expression

$$\alpha = dd^c(d^* G_d (d^c)^* G_{d^c} \alpha)$$

and we define the thing in perentheses to be β. □

Definition. Define a cochain complex

$$i : \Omega^\bullet_{d^c-cl}(M), d) \hookrightarrow (\Omega^\bullet(M), d)$$

consisting of d^c-closed forms.

Lemma. i is a quasi-isomorphism.

Proof. (surjective) Suppose $\alpha \in \Omega_{cl}(M)$ and $\beta = d^c\alpha$. Then we have

$$d\beta dd^c\alpha = -d^c d\alpha = 0$$

so we can apply the dd^c-lemma to see

$$d\beta = dd^c\gamma$$

and therefore

$$d^c(\alpha + d\gamma) = \beta - dd^c\gamma = 0.$$

(injective) similar application of the main lemma. □

We now want to show that this new cochain complex is quasi-isomorphic to a complex with zero differential. **Definition.** Define

$$(\overline{\Omega}_{d^c-cl}, \overline{d}) = \Omega_{d^c-cl}(M)/(imd^c).$$

Lemma. $\overline{d} = 0$. The map

$$p : \Omega^\bullet_{d^c-cl}(M) \to \overline{Omega}^\bullet_{d^c-cl}(M)$$

is a quasi-isomorphism.

"Proof of $\overline{d} = 0$." Take $\alpha \in \Omega_{d^c-cl}(M)$ and $\beta = d\alpha$. Then we have

$$d^c\beta = d^c d\alpha = 0$$

so by the dd^c-lemma, we can write

$$\beta = -d^c d\gamma$$

which implies $\overline{d}[\alpha] = 0$. $\square$

Main Theorem.

1. Kähler manifolds are formal
2. If $f : M \to N$ is a map of compact Kähler manifolds, then $f^* \simeq H^* f^*$.

Applications. All Massey products vanish for the following: (which are Kähler manifolds)

1. Projective varieties
2. $\mathbb{CP}^n$, $\mathbb{C}^n$
3. Stein manifolds
4. K3 surfaces
5. Blow-ups, covers, certain quotients by properly discontinuous isometric group actions
6. Complex submanifolds of these things
7. Products of these things

Lemma. "Iwasawa manifolds" (even betti numbers) have higher Massey products

Corollary. Iwasawa manifolds are not Kähler