Rational Homotopy Theory Seminar

Week 11: Obstruction theory for rational homotopy equivalences

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Reference. Halperin-Stasheff "Obstructions to homotopy equivalences"

Question. When can a given isomorphism $H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ be realized by a continuous map $X \to Y$? This question is quite difficult, so instead we can ask the following:

When can a given isomorphism $f : H_*(X;\mathbb{Q}) \xrightarrow{\cong} H^*(Y;\mathbb{Q})$ of rational cohomology algebras be realized by a rational homotopy equivalence between X and Y? Note here that a rational equivalence is a zig-zag of "elementary equivalences", i.e. maps inducing isomorphisms in rational cohomology.

A space X is *rationally nilpotent* if its Sullivan minimal model $A_X = (\Lambda V, d) \simeq \Omega^*_{poly}(S_\bullet(X))$ has only finitely many generators in each degree. Here, ΛV is the free commutative graded algebra on V. The relevance of this definition is seen through the following example and main theorem.

Example. Any nilpotent path-connected space with finite dimensional rational homology in each degree is rationally nilpotent. A space X is called *nilpotent* if $\pi_1(X)$ is a nilpotent group and if $\pi_1(X)$ acts nilpotently on the higher homotopy groups of X. In particular, any simply-connected space with finite-dimensional rational homology in each degree is nilpotent.

Theorem 1.3. Assume X, Y are rationally nilpotent. Then the isomorphism $f : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ can be realized by a rational homotopy equivalence if and only if the obstructions $O_n(f)$ all vanish.

Note that if the Sullivan minimal models $A_X \cong A_Y$ are isomorphic, then X and Y are rationally homotopy equivalent. The theorem says that showing the vanishing of the obstructions $O_n(f)$ is another way of exhibiting this rational homotopy equivalence. We'll conclude with an example of why this would be desirable, and then leave the applications to formality and CDGA's over field extensions (covered in Section 6 of the paper) for the exercises.

By the chain of Quillen equivalences discussed in PJ's minicourse, we can rephrase everything in terms of homotopy equivalences (zig-zags of maps inducing isos in homology) between CDGA's. So, suppose we have CDGA's A and B and a fixed isomorphism $f : H(A) \xrightarrow{\cong} H(B)$, and assume further that $H(A)$ is connected and has finite type.

Theorem 5.10. The isomorphism f can be realized by a homotopy equivalence if and only if the sequence $O_n(f)$ vanish.

Using the Quillen equivalences from the minicourse, it is not hard to show that Theorem 5.10 implies Theorem 1.3.

We begin with some preliminary definitions. A *connected Koszul-Sullivan complex* is a CDGA of the form $(\Lambda X, D)$ where $X = \Sigma_{p>0} X^p$ is a strictly positive graded space and D

satisfies the *nilpotence condition* that there is a homogeneous basis $\{x_\alpha\}_{\alpha\in\zeta}$ for X where ζ is a well-ordered set, such that Dx_α is a polynomial in the x_β with $\beta < \alpha$. A Koszul-Sullivan complex is *minimal* if $D(X) \subset (\Lambda^+ X) \cdot (\Lambda^+ X)$.

The path-complex of a connected K-S complex $(\Lambda X, D)$, denoted by $(\Lambda X, D)^I$, is the CDGA $(\Lambda X \otimes \Lambda \overline{X} \otimes \Lambda \hat{X}, \overline{D})$ where

1. $\overline{D}|_{\Lambda X} = D$
2. $\overline{X}$ is the graded space defined by $\overline{X}^p = X^{p+1}$
3. $\hat{X}$ is a graded space isomorphic with X
4. $\overline{D}\overline{x} = \hat{x}$ and $\overline{D}\hat{x} = 0$.

These conditions uniquely determine $\overline{D}$. A *homotopy* between CDGA maps $\phi_0, \phi_1 : (\Lambda X, D) \to (A, d_A)$ is a map $\Phi : (\Lambda X, D)^I \to (A, d_A)$ satisfying the expected relations.

Using this definition, one can prove that given a quasi-isomorphism $\phi : (A, d_A) \to (B, d_B)$ and a map $\psi : (\Lambda X, D) \to (B, d_B)$ with $(\Lambda X, D)$ connected and nilpotent, then there exists a unique (up to homotopy) homomorphism $\chi : (\Lambda X, D) \to (A, d_A)$ such that $\phi \circ \chi \simeq \psi$.

We say that a CDGA (A, d_A) is *c-connected* if $H(A)$ is connected. We have already discussed minimal Sullivan models in previous weeks; the following theorem says that if in addition the CDGA you start with is c-connected, then the resulting minimal model can be required to be nilpotent.

Theorem 2.6. Let (A, d_A) be a c-connected CDGA. Then there is a minimal connected Koszul-Sullivan complex (M_A, δ_A) and a homomorphism $m_A : (M_A, \delta_A) \to (A, d_A)$ such that m_A^* is an isomorphism. The resulting complex is unique up to homotopy.

In particular, a homomorphism between minimal connected K-S complexes is an isomorphism if and only if it induces an isomorphism of cohomology.

We say that $m_A : (M_A, \delta_A) \to (A, d_A)$ is the *minimal model* for (A, d_A). A *special homotopy equivalence* between CDGA's (A, d_A) and (B, d_B) is a homotopy equivalence between their minimal models M_A and M_B; such an equivalence gives rise to a homotopy equivalence between A and B. Conversely, given a homotopy equivalence between A and B, one can define an obvious special homotopy equivalence between their minimal models. This is summarized in the following propositions.

Propositions 2.10, 2.11. An isomorphism $f : H(A) \xrightarrow{\cong} H(B)$ can be realized by a homotopy equivalence if and only if there is an isomorphism $\phi : (M_A, \delta_A) \xrightarrow{\cong} (M_B, \delta_B)$ such that $f = m_B^* \circ \phi^* \circ (m_A^*)^{-1}$.

1. If $f : H(A) \to H(B)$ and $g : H(B) \to H(c)$ are realizable isomorphisms, then $g \circ f$ and f^{-1} are realizable.

2. If $G(A, d_A)$ is the group of realizable automorphisms of $H(A)$ and $f : H(A) \overset{\cong}{\to} H(B)$ is realizable, then the group isomorphism $G(A, d_A) \overset{\cong}{\to} G(B, d_B)$ is given by $g \mapsto f \circ g \circ f^{-1}$.

We now turn to the first key construction of the paper. Beginning with a connected CGA H, one obtains a *bigraded* model as follows. Regard H as a CDGA with trivial differential; by the above, it has a minimal model

$$\rho : (\Lambda Z, d) \to (H, 0).$$

We'll explicitly construct this minimal model by defining a sequence of graded spaces $Z_0, Z_1, \ldots$ such that $Z = \Sigma_{n=0}^{\infty} Z_n$. Denote by $Z_{(n)} = Z_0 \oplus \cdots \oplus Z_n$. We'll then define ρ and d so that

1. $\rho : \Lambda Z_0 \to H$ is surjective
2. $\rho^* : H_0(\Lambda Z_{(1)}, d) \overset{\cong}{\to} H$
3. $\rho^* : H_0(\Lambda Z_{(n)}, d) \overset{\cong}{\to} H$ and $H_i(\Lambda Z_{(n)}, d) = 0$ for $1 \leq i < n$ and $n \geq 2$

We'll also write $Z_n^p = Z^{-n, p+n}$ and $(\Lambda Z)_n^p = (\Lambda Z)^{-n, p+n}$.

Construction. The space $Z_0 = H^+ / (H^+ \cdot H^+)$ is the space of indecomposables for H. Set $d = 0$ in Z_0, and define $\rho : \Lambda Z_0 \to H$ so its restriction to Z_0 splits the projection $H^+ \to Z_0$. These are the "generators" for H. Then ρ is surjective with kernel K satisfying $K^0 = K^1 = 0$.

The space $Z_1 = K/(K \cdot \Lambda^+ Z_0)[1]$, i.e. $Z_1^p = (K/K \cdot \Lambda^+ Z_0)^{p+1}$. These are the "relations" for H. Since $K^0 = K^1 = 0$, we have $Z_1 = \Sigma_{p \geq 1} Z_1^p$. Extend d to Z_1 by requiring that it be a linear map $Z_1 \to K$ splitting the projection. Then d is homogeneous of lower degree -1 in $\Lambda Z_{(1)}$. Extend ρ to be zero on Z_1.

In general, the space Z_n is defined to kill off H_n. If we've constructed Z_n already, define Z_{n+1} by

$$Z_{n+1}^p = [H_n(\Lambda Z_{(n)}, d) / (H_n(\Lambda Z_{(n)}, d) \cdot H_0^+(\Lambda Z_{(n)}, d))]^{p+1},$$

then extend d so that $d : Z_{n+1} \to (\Omega Z_{(n)})_n \cap ker d$ splits the projection onto Z_{n+1} and extend ρ to be zero on Z_{n+1}.

Proposition 3.4. The CDGA $(\Lambda Z, d)$ satisfies

1. $\rho^* : H_0(\Lambda Z, d) \overset{\cong}{\to} H$
2. $H_{\geq 1}(\Lambda Z, d) = 0$
3. $(\Lambda Z, d) \xrightarrow{\rho} (H, 0)$ is a minimal model.

It is the unique bigraded algebra with differential of "lower degree" -1 satisfying these properties up to isomorphism.

The proof is by induction on n and from the definition, so the interested reader is referred to the paper. It's interesting to note that

$$\cdots \xrightarrow{d} (\Lambda Z)_{n+1} \xrightarrow{d} (\Lambda Z)_n \xrightarrow{d} \cdots \xrightarrow{d} (\Lambda Z)_0 = \Lambda Z_0 \xrightarrow{\rho} H$$

is a resolution of H by free ΛZ_0-modules, so it can be used to calculate $Tor_{\Lambda Z_0}(H, -)$.

Exercise. Let $H = \Lambda(x_1, \ldots, x_4)/I$ where $|x_1| = |x_2| = |x_3| = 3$ and $|x_4| = 5$ and $I = \langle x_1x_2, x_1x_3x_4, x_2x_3x_4\rangle$, and consider H as an DGA with trivial differential $(H, 0)$.
Determine bases and differentials for Z_0, Z_1, Z_2.

Remark. If H has finite type, then one can compute the Poincare series for H, $\sum_{p=0}^{\infty}(dimH^p)t^p$ using the integers $dimZ_n^2$. We'll leave the derivation of this formula to the exercises.

We now construct the canonical filtered model for a c-connected CDGA (A, d_A) by perturbing the bigraded model $(\Lambda Z, d) \xrightarrow{\rho} (H(A), 0)$. Define an increasing filtration of ΛZ by

$$F_n(\Lambda Z) = \Sigma_{m \leq n}(\Lambda Z)_m.$$

A linear map $\phi : \Lambda Z \to \Lambda Z$ is *filtration decreasing* if

$$\phi(F_n(\Lambda Z)) \subset F_{n-1}(\Lambda Z)$$

for each n. If ϕ is a derivation, this is the same as saying that

$$\phi(Z_n) \subset F_{n-1}(\Lambda Z).$$

The idea is to model (A, d_A) by a CDGA $(\Lambda Z, D)$ such that

$$(D - d) : Z_n \to F_{n-2}(\Lambda Z)$$

where D is a perturbation of d in that $D = d_1 + d_2 + \ldots$ where $d_i|_{Z_n} \subset (\Lambda Z)_{n-i}$. In other words, a perturbation is a map which sends elements to elements of strictly lower filtration.

Example. In the previous exercise, one shows that Z_2 has a basis $\{z_1, \ldots, z_{10}\}$ with differentials $dz_1 = y_1x_1$ and $dz_i = else$ for $i = 2, \ldots, 10$. Let's perturb $(\Lambda Z, d)$ to a CDGA $(\Lambda Z, D)$ so that $D - d : Z_n \to F_{n-2}(\Lambda Z)$. One is forced to define $D = d$ on Z_0 and Z_1, but on Z_2 one defines $Dz_1 = y_1x_1 + x_3x_4$ and $Dz_i = dz_i$ for $i = 2, \ldots, 10$. Then $D^2 = 0$ on $\Lambda Z_{(2)}$. Induction shows D can be extended to all of ΛZ. In particular, one shows that $Dw = z_1x_1 - y_2$ for some $w \in Z_3$, so that $(\Lambda Z, D)$ is not minimal and cannot be isomorphic to $(\Lambda Z, d)$.

Theorem 4.4. Let (A, d_A) be a c-connected CDGA and let $\rho : (\Lambda Z, d) \to (H(A), 0)$ be the bigraded model for $H(A)$. Then there is a CDGA $(\Lambda Z, D)$ and a homomorphism $\pi : (\Lambda Z, D) \to (A, d_A)$ such that

1. $(D - d) : Z_n \to F_{n-2}(\Lambda Z), \quad n \geq 0$

2. $[\pi z] = \rho z$ for $z \in \Lambda Z_0$

3. π^* is an isomorphism

Moreover, suppose $\pi' : (\Lambda Z, D') \to (A, d_A)$ satisfies the same conditions. Then there is an isomorphism $\phi : (\Lambda Z, D) \xrightarrow{\cong} (\Lambda Z, D')$ such that

1. $(\phi - \iota)$ is filtration decreasing

2. $\pi'\phi \simeq \pi : (\Lambda Z, D) \to (A, d_A)$.

Sketch of proof. (Existence) One constructs D and π inductively on $Z_0, Z_1, \ldots$ as follows. Fix a splitting $\eta : H(A) \to \Lambda Z_0$ of ρ. For Z_0, Z_1, Z_2, the definitions of D and π are forced by the choice of splitting. Assuming one has extended the definitions to $\Lambda Z_{(n)}$ for some $n \geq 2$, the $(n+1)$-case uses some clever definitions and linear algebra, so it's best read in the paper.

(Uniqueness) One must write an explicit homotopy between $\pi : (\Lambda Z, D) \to (A, d_A)$ and $\pi' : (\Lambda Z, D') \to (A, d_A)$. This involves a rather long induction argument, but as above, one makes the correct definitions at each step so that the correct notion of uniqueness is satisfied at each step. □

Recap. Starting with a CDGA (A, d_A), we view its homology $H(A)$ as a CDGA $(H(A), 0)$, then produce a bigraded model $(\Lambda Z, d) \xrightarrow{\rho} (H(A), 0)$, and finally we perturb the bigraded model to obtain an isomorphic bigraded model $(\Lambda Z, D) \xrightarrow{\pi} (A, d_A)$. The last bigraded model is called the *filtered model* for (A, d_A).

Obstruction theory. Fix an isomorphism of graded algebras

$$f : H(A) \to H(B)$$

where (A, d_A) and (B, d_B) are c-connected CDGA's. Let $\rho_A : (\Lambda Z, d) \to (H(A), 0)$ be the bigraded model. By uniqueness, the composition $\rho_B = f \circ \rho_A : (\Lambda Z, d) \to (H(B), 0)$ is the bigraded model for $H(B)$. Fix linear maps $\eta_A : H(A) \to \Lambda Z_0$ and $\eta_B : H(B) \to \Lambda Z_0$ so that composition with the bigraded model structure maps are the inclusion.

Perturb both bigraded models as above to obtain

$$\pi_A : (\Lambda Z, D_A) \to (A, d_A)$$

$$\pi_B : (\Lambda Z, D_B) \to (B, d_B)$$

Theorem 5.3. The map f can be realized by a homotopy equivalence if and only if there is an isomorphism $\phi : (\Lambda Z, D_A) \to (\Lambda Z, D_B)$ such that $\phi - \iota$ decreases filtrations.

Proof. ($\Leftarrow$) Given such a ϕ, the sequence π_A, ϕ, π_B is a special homotopy equivalence, so by remarks from above, gives rise to an actual realization of f. To see that the sequence is a special homotopy equivalence, proceed as follows. If $\alpha \in H(A)$, then $\eta_A \alpha \in \Lambda Z_0$ satisfies $[\pi_A \eta_A \alpha] = \alpha$ by the construction of Z_0. Since $\phi_{\Lambda Z_0} = id$, we have

$$\pi_B^* \circ \phi^* \circ (\pi_A^*)^{-1}(\alpha) = [\pi_B(\eta_A \alpha)] = \rho_B \eta_A \alpha = f \rho_a \eta_A \alpha = f(\alpha)$$

so the sequence is indeed a special homotopy equivalence.

($\Rightarrow$) If f can be realized by a homotopy equivalence, the above remarks imply there is an isomorphism

$$\psi : (M_A, \delta_A) \to (M_B, \delta_B)$$

between the associated minimal models such that $m_B^* \circ \psi^* \circ (m_A^*)^{-1} = f$. By uniqueness of nilpotent models, there is a homomorphism

$$\gamma : (\Lambda Z, D_A) \to (M_A, \delta_A)$$

such that $m_A \circ \gamma \simeq \pi_A$. One can verify that $m_B \circ \psi \circ \gamma : (\Lambda Z, D_A) \to (B, d_B)$ satisfies the conditions of Theorem 4.4, so there is an isomorphism $\phi : (\Lambda Z, D_A) \xrightarrow{\cong} (\Lambda Z, D_B)$ such that $\phi - \iota$ decreases filtrations. $\square$

Definition. The isomorphism $f : H(A) \xrightarrow{\cong} H(B)$ is *n-realizable* if there is an isomorphism $\phi : (\Lambda Z_{(n+1)}, D_A) \to (\Lambda Z_{(n+1)}, D_B)$ such that $\phi - \iota$ decreases filtrations. In this case, ϕ is called an *n-realizer* for f.

If ϕ is an n-realizer for f, the degree 1 linear map

$$o(\phi) : Z_{n+2} \to H(B)$$

$$z \mapsto [\pi_B \phi D_A z]$$

is called the *obstruction element determined by* ϕ. The set of these is denoted

$$O_{n+1}(f) = \{o(\phi) : \phi \text{ is an } n\text{-realizer for } f\}.$$

It's clear that if f is realizable, then it is n-realizable for all n. We'll address the converse after we finish setting up the obstruction theory. Note that

$$O_{n+1}(f) \subset Hom^1(Z_{n+2}, H(B))$$

i.e. degree 1 maps from Z_{n+2} to $H(B)$. Let

$$M_n \subset Der(\Lambda Z_{(n)})$$

be the space of filtration decreasing derivations θ of degree zero in $\Lambda Z_{(n)}$ which commute with the filtered model differential D_B, i.e. $D_B \theta = \theta D_B$. Define a linear map

$$\gamma : M_n \to Hom^1(Z_{n+1}, H(B))$$

$$\gamma(\theta)(z) = [\pi_B \theta D_B z].$$

With all of this in place, we define the obstructions as follows:

Proposition 5.6 Suppose ϕ is some $(n-1)$-realizer for f. Then

$$O_n(f) = o(\phi) + \gamma(M_n).$$

Proof. Note that any other $(n-1)$-realizer ϕ' for f is related to ϕ by some automorphism ψ of $(\Lambda Z_{(n)}, D_B)$ such that $\psi - \iota$ decreases filtration. One can verify that any such automorphism is of the form $\psi = e^\theta = \sum_{p=0}^{\infty}(1/p!)\theta^p$ where $\theta \in M_n$, so the $(n-1)$-realizers of f are the isomorphisms of the form $e^\theta\phi$, $\theta \in M_n$.

Therefore $O_n(f) = \{o(e^\theta\phi) : \theta \in M_n\}$ and we need to show

$$o(e^\theta\phi) = o(\phi) + \gamma(\theta).$$

This is somewhat involved, but is not hard to follow and is left to the interested reader. □

Some consequences of the "involved part" of the previous proof are the following results:

Proposition 5.7, Corollary 5.8. An $(n-1)$-realizer ϕ for f extends to an n-realizer if and only if $o(\phi) = 0$. In particular, if f is $(n-1)$-realizable, then f is n-realizable if and only if

$$O_n(f) = \gamma(M_n).$$

Therefore $O_n(f)$ may be regarded as a single element in $Hom^1(Z_{n+1}, H(B))/\gamma(M_n)$, and it's this single element that we think of as the obstruction. We can now address the converse:

Theorem 5.10. If $H(A)$ has finite type, then f can be realized by a homotopy equivalence if and only if all the obstruction classes $O_n(f)$ vanish.

Theorem 5.15. Assume $H^p(A) = 0$ for $1 \leq p \leq l$ and for $p > m$. Then f is realizable by a homotopy equivalence if and only if

$$O_n(f) = 0, \quad 1 \leq n \leq \frac{m-2}{l} - 2.$$

Proof. The forward implication is obvious from Theorem 5.3, so we need to prove the reverse implication. Suppose $O_n(f) = 0$ for $n \leq (m-2)/l - 2$, let N be the largest integer n, and let ϕ be an N-realizer for f. We have

$$O(\phi) \in Hom^1(Z_{N+2}, H(B)).$$

One can show by induction on k that if $H^p(A) = 0$ for $1 \leq p \leq l$, then $Z_k^p = 0$ for $1 \leq p \leq (k+1)l$. Since $N+1 > (m-2)/l - 2$, $H^p(A) \cong H^p(B) = 0$ for $p > m$, we have $Hom^1(Z_{N+2}, H(B)) = 0$. Therefore $o(\phi) = 0$ and ϕ extends to an $(N+1)$-realizer ϕ_1 for f.

Repeating this process gives a sequence of $(N+k)$-realizers ϕ_k for f, and one then defines $\phi : (\Lambda Z, D_A) \to (\Lambda Z, d_B)$ by setting $\phi(u) = \phi_p(u)$ for $u \in \Lambda Z_{(N+p+1)}$. This satisfies Theorem 5.3, so f is realizable. □

The proof of Theorem 5.10 is similar; one cleverly chooses a sequence of integers $m_1 \leq m_2 \leq \cdots$ and the sequence of m_n-realizers ϕ_n piece together as above into an isomorphism satisfying Theorem 5.3, and therefore give rise to a homotopy equivalence.