Rational Homotopy Theory Seminar

Week 2: Minicourse Part II: Attack of the Quillen Equivalent Categories

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Recall that we defined the rationalization $X \to X_{\mathbb{Q}}$ with X simply connected to satisfy $\pi_k(X_{\mathbb{Q}}) = \pi_k(X) \otimes \mathbb{Q}$. We also saw that for X finite CW, the homotopy groups split $\pi_k(X) = \mathbb{Z}^r \oplus T$ into a free and torsion part. We also saw that $\pi_*(S^n_{\mathbb{Q}})$ is a free graded Whitehead algebra. We wanted to show

$$Top \rightleftarrows sSet \rightleftarrows CDGA \rightleftarrows CDGC \rightleftarrows DGL.$$

Let's give some names. We have

$$S_\bullet : Top \rightleftarrows sSet : |-|$$

and

$$\Omega^\bullet_{poly} : sSet \rightleftarrows CDGA :< - >$$

and

$$(-)^\vee : CDGA \rightleftarrows CDGC : (-)^\vee$$

and

$$\mathcal{L} : CDGC \rightleftarrows DGL : \mathcal{C}.$$

A *model category* is the kind of category where we can do homotopy theory. More precisely, a model category is a category C equipped with three distinguished classes of morphisms

- (weak equivalences) $\xrightarrow{\sim}$
- (cofibrations) $\rightarrowtail$
- (fibrations) $\twoheadrightarrow$

satisfying

1. All limits and colimits exist in C
2. If any two of $f, g, g \circ f$ in $X \xrightarrow{f} Y \xrightarrow{g} Z$ are weak equivalences, then so is the third
3. Retracts of weak equivalences, cofibrations, fibers are weak equivalences, cofibrations, fibrations
4. Lifting properties for cofibration, acyclic fibration and fibration, acyclic cofibration
5. Every map $f : X \to Y$ factors as $X \rightarrowtail Z \overset{\sim}{\twoheadrightarrow} Y$ and as $X \overset{\sim}{\rightarrowtail} Z' \twoheadrightarrow Y$

For example, let $C = Top$. Weak equivalences are weak homotopy equivalences, fibrations are Serre fibrations, and cofibrations are morphisms which satisfy the left lifting property with respect to acyclic fibrations, i.e. retracts of relative cell complexes

Note that you can have multiple different model structures. Also note that if C is a model category, there is automatically a model structure on C^{op}.

The other categories in the chain of Quillen equivalences are also model categories, as is the category of connected chain complexes over R, denoted $Ch^R_{\geq 0}$. The weak equivalences in the above categories are:

- (CDGA) homology isomorphisms
- (DGL) homology isomorphisms

The *homotopy category* of a model category C, denoted $Ho(C)$, is the category $C[we^{-1}]$ where we've formally inverted all weak equivalences.

A *Quillen adjunction* between model categories C and D is an adjunction $F : C \rightleftarrows D : G$ such that F preserves cofibrations and G preserves fibrations. Each Quillen adjunction induces an adjunction on homotopy categories.

A *Quillen equivalence* is a Quillen adjunction $F : C \rightleftarrows D : G$ which induces an equivalence of homotopy categories $LF : Ho(C) \rightleftarrows Ho(D) : RG$. In other words, we have $RG(LF(X)) \simeq X$.

The restrictions we need on our categories of interest are expressed below. We claim that each of these are Quillen equivalences.

$$Top \rightleftarrows sSet$$

$$sSet^{fin,\geq 1}_{\mathbb{Q}} \rightleftarrows CDGA^{fin,\geq 1}_{\mathbb{Q}}$$

$$CDGA^{fin} \rightleftarrows CDGC^{fin}$$

$$CDGC^{\geq 1} \rightleftarrows DGL^{conn,\geq 2}$$

A *simplicial object* in a category C is a functor $X : \Delta^{op} \to C$ where Δ is the category of natural numbers $[n] = \{0, 1, \ldots, n\}$ and nondecreasing maps.

Alternatively, we can think of X as a sequence of objects $X_0, X_1, \ldots$ in C with simplicial structure maps between them.

The simplicial chains functor $S_\bullet : Top \to sSet$ is defined by sending $X \to Hom_{Top}(\Delta^n, X) =: S_n(X)$ with structure maps $d_i : S_n(X) \to S_{n-1}(X)$ defined by $d_i(f)(t_0, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n$ and $s_i : S_n(X) \to S_{n+1}(X)$ defined by $s_i(F)(t_0, \ldots, t_n) = f(t_0, \ldots, t_i + t_{i+1}, \ldots, t_n)$.

The geometric realization functor $|-| : sSet \to Top$ sends $S \mapsto |S| = \sqcup_{n>0}(S_n \times \Delta^n)/\sim$ where $(d_i(x), t) \sim (x, d^i(t))$ and $(s_i(x), t) \sim (x, s^i(t))$.

It turns out that these form an adjuntion since we can show

$$Top(|S|, X) \cong sSet(S, S_\bullet(X)).$$

This follows from

Theorem. For all $X \in Top$, we have a weak homotopy equivalence $|S_\bullet(X)| \simeq X$.

A *differential graded algebra*, or *cochain algebra*, over k is a sequence of k-modules

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \cdots$$

with a multiplication map $\mu : A^p \otimes A^q \to A^{p+q}$ and unit $1 \in A^0$ such that

1. $(xy)z = x(yz)$
2. $1x = x1 = x$
3. $d^2 = 0$
4. $d(xy) = d(x)y + (-1)^{|x|}xd(y)$.

We say that it is *commutative* if $xy = (-1)^{|x||y|}yx$. A *morphism* $g : A^* \to B^*$ consists of maps $g^n : A^n \to B^n$ which are k-linear, preserve the unit, and commute with d and μ.

Recall that we have

$$X \mapsto \pi_* X \mapsto s\pi_* X$$

$$Top \to WhiteheadAlgs \to GrLieAlgs$$

and we can also consider

$$X \mapsto S_\bullet X \mapsto C^*(S_\bullet X)$$

$$Top \to sSet \to DGA_{\geq 0}$$

where $C^n(S,k) = \{f : S_n \to k : f(degenerate) = 0\}$ with $d : C^n(S,k) \to C^{n+1}(S,k)$ defined by $d(f)(x) = \sum_{i=0}^{n+1}(-1)^{1+n} f(d_i(x))$. One can also define a cup product

$$\cup : C^p(S,k) \otimes C^q(S,k) \to C^{p+q}(S,k)$$

defined by

$$(f \cup g)(x) = f(x_{0\ldots p}g(x_{p\ldots p+q}).$$

Commutative cochain problem. We want to find a functor $A^* : sSet \to CDGA$ such that $A^*(S) \simeq C^*(S,k)$. It turns out that this isn't possible for a general k, but Sullivan was able to use the functor $\Omega^\bullet_{poly}$ for rings k containing $\mathbb{Q}$.