Rational Homotopy Theory Seminar
Week 3: Minicourse Part III
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Last time we talked about the *Commutative Cochain Problem*, i.e. we want a functor $\mathcal{A} : sSet \to CDGA_k$ such that its image $\mathcal{A}^*(X) \simeq C^*(X;k)$. Note that if we have $A \in DGA$, its cohomology $H^*(A)$ is a differential graded algebra. Here we are assuming k is a field of characteristic zero.

Theorem. The category of commutative differential graded algebras over k, $CDGA_k$, is a model category. The weak equivalences are quasi-isomorphisms (i.e. isomorphisms in cohomology), the fibrations are (levelwise) surjective morphisms, and the cofibrations are maps which satisfy the left lifting property with respect to acyclic fibrations.

We'll show that the *polynomial differential forms functor* $\Omega^*_{poly} : sSet \to CDGA$ satisfies the commutative cochain problem. How do we define this functor?

Let $A^*_\bullet \in sDGA$, i.e. $A^*_\bullet : \Delta^{op} \to DGA$. Define $A^* : sSet \to DGA$ be the functor uniquely characterized by

1. $A^*(\Delta[n]) = A^*_n$
2. A^* takes colimits to limits

It is defined explicitly by $A^p(X) = sSet(X, A^p_\bullet)$. To define the differential, let $f \in sSet(X, A^p_\bullet)$. Then df is defined by $(df)(x) = d(f(x))$. The product structure is defined by $(fg)(x) = f(x)g(x)$. The unit is defined by $1(x) = 1$. Note that the properties above also show that A^* is a Kan extension.

Since $\Omega^*_\bullet \in sDGA$, each level Ω^*_n is a DGA. The *polynomial differential forms on an n-simplex* is defined as

$$\Omega^*_n = \frac{k[t_0, \dots, t_n] \otimes \Omega(dt_0, \dots, dt_n)}{(t_0 + \cdots + t_n - 1, dt_0 + \cdots + dt_n)}$$

where $|t_i| = 0$ and $|dt_i| = 1$. The differential $d : \Omega^*_n \to \Omega^{*+1}_n$ is given by $d(f) = \sum_{i=0}^n \frac{\partial f}{\partial t_i} dt$.

The simplicial structure on $\Omega^*_\bullet$ is defined as follows. Let $\phi \in \Delta([m],[n])$. Then $\phi^* \in DGA(\Omega^*_n, \Omega^*_m)$ is defined by $\phi^*(t_i) = \sum_{j \in \phi^{-1}(i)} t_j$ and then the Leibniz rule.

Applying the above, we have $\Omega^i_{poly}(X) = sSet(X, \Omega^i_\bullet)$ with $(df)(x) = d(f(x))$, $(fg)(x) = f(x)g(x)$, and $1(x) = 1$. This functor $\Omega^*_{poly} : sSet \to CDGA$ will satisfy the commutative cochain problem after adding some hypotheses.

Theorem. If $k \supseteq \mathbb{Q}$, then $\Omega^*_{poly} \xrightarrow{\sim} Z \xleftarrow{\sim} C^*(X;k)$.

What is the adjoint of this functor? We mentioned previously that it is *spatial realization*, denoted $< - >: CDGA \to sSet$. Let $A \in CDGA$. Then define $< A >= DGA(A, \Omega^*_\bullet)$, so for example $< A >_n = DGA(A, \Omega^*_n)$. The simplicial structure is defined as follows. If $\phi \in A([m],[n])$, then $\tilde{\phi} \in DGA(< A >_n, < A >_m)$ is defined by $\tilde{\phi}(f) = \phi^* \circ f$.

These functors form a contravariant adjuntion, and in fact, they form a Quillen equivalence:

Theorem. The derived functors (denoted $D(-)$ below) of the adjunction form an equivalence of homotopy categories

$$D < - >: Ho(CDGA_{\mathbb{Q}}^{\geq 1, fin}) \rightleftarrows Ho(sSet_{\mathbb{Q}}^{\geq 1, fin}) : D\Omega^*_{poly}.$$

Sullivan algebras are the main examples of commutative differential graded algebras we are concerned with. Comparing Top and $CDGA$, we have the analogy that CW-complexes correspond to Sullivan algebras. Furthermore, just as every space has a CW-approximation, every connected DGA has a Sullivan model.

The analogs of spheres S^n and disks D^n are denoted $S(n)$ and $D(n-1)$. They are defined as follows. For $n > 0$define

$$S(n) = (\Lambda x, dx = 0)$$

and

$$D(n-1) = (\Lambda(x, sx), dx = 0, d(sx) = x)$$

where $|x| = n$ in both definitions and $\Lambda = Sym$ is the free functor. Note that $S(n) \subset D(n-1)$. In fact, just as the generating cofibrations of Top are inclusions $S^n \hookrightarrow D^{n+1}$, the generating cofibrations in $CDGA$ are the inclusions $S(n) \hookrightarrow D(n-1)$.

If V is a graded vector space, we can define

$$S(V) = (\Lambda(V), dv = 0)$$

and

$$D(V) = (\Lambda(V \oplus sV), dv = 0, d(sv) = v).$$

In general, a *Sullivan algebra* is defined as follows. Start with k in degree 0. Add (positive degree) free generators with zero differential. For example, the sphere $S(n)$ is obtained by adding one generator in each degree a multiple of n. We can also add relations: add (positive degree) free generators to kill the initial generators.

A *minimal Sullivan algebra* is a Sullivan algebra where the differential d only kills elements of word length at least two. For example, the sphere is minimal but $D(n-1)$ is not minimal since the differential kills x which has word length one.

Theorem. Let $A \in CDGA^{\geq 1}$, i.e. $H^0(A) = k$ and $H^1(A) = 0$. Then A admits a unique (up to isomorphism of CDGA's) minimal Sullivan model, i.e. a map from a minimal Sullivan algebra to A

$$(\Lambda V, d) \xrightarrow{\sim} A$$

inducing an isomorphism on cohomology.

If $X \in Top_{\mathbb{Q}}^{\geq 1, fin}$, then its *minimal Sullivan model* is

$$A_X \xrightarrow{\sim} \Omega^*_{poly}(S_\bullet(X)).$$

Corollary. If $X \in Top_{\mathbb{Q}}^{\geq 1, fin}$, then

1. X has a unique, up to isomorphism, minimal Sullivan model $A_X = (\Lambda V, d)$ where $V = V^{\geq 2}$ is a finitely generated $\mathbb{Q}$-vector space.

2. If $X, Y \in Top_{\mathbb{Q}}^{\geq 1, fin}$, then $X \simeq_{\mathbb{Q}} Y \iff A_X \cong A_Y$.

3. We can compute the rational homotopy groups of X as the dual below:

$$\pi_k(X) \otimes \mathbb{Q} \cong \left(\frac{A_X^{\geq 0}}{A_X^{>0} \cdot A_X^{>0}} \right)^{\vee} = (V)^{\vee}$$

The main facts we will use are:

1. $H^*(X; \mathbb{Q}) \cong H^*(A_X)$

2. $\pi_*(X) \otimes \mathbb{Q} \cong (V)^{\vee}$

Example. If $K(\mathbb{Q}, n)$ is an Eilenberg-Mac Lane space, then $\pi_n(K(\mathbb{Q}, n)) \cong \mathbb{Q}$ and $\pi_i(K(\mathbb{Q}, n)) = 0$ for $i \neq 0$. Using (2) above, the corresponding CDGA is $S(n)$. Therefore $H^*(K(\mathbb{Q}, n); \mathbb{Q}) \cong H^*(S(n)) = Sym(x)$ with $|x| = n$. If n is even, $Sym(x) = P(x)$ and if n is odd, $Sym(x) = E(x)$.