Rational Homotopy Theory Seminar
Week 4: Minicourse Part IV
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Recall. If $X \in Top_{\mathbb{Q}}^{\geq 1, fin}$, then there exists a unique (up to isomorphism) minimal Sullivan model

$$A_X = (\Lambda V, d) \xrightarrow{\simeq} \Omega^*_{poly}(S_\bullet(X)).$$

We saw an isomorphism of commutative differential graded algebras $H^*(X;\mathbb{Q}) \cong H^*(A_X)$, and we also saw $\pi_*(X) \otimes \mathbb{Q} \cong (V)^\vee$ where V was the indecomposables in $A_X^{\geq 0}$.

A Sullivan algebra is the differential free graded algebra defined by putting k in degree 0 and then adding generators in higher degrees with zero differential, then killing some of those elements via the differential on new generators. This was denoted $(\Lambda V, d)$. We said it was a minimal Sullivan algebra if $d(V) \subset \Lambda^{\geq 2}(V)$.

Examples. We want to understand the correspondence between $Top_{\mathbb{Q}}^{\geq 1, fin} \leftrightarrow CDGA_{\mathbb{Q}}^{\geq 1, fin}$.

- If X is a space, then it maps under this correspondence to A_X.
- If we start with $K(\mathbb{Q}, n)$, then it maps to $S(n) = \Lambda(x)$ with $|x| = n$. To see this, note that we have $H^*(K(\mathbb{Q}, n);\mathbb{Q}) = \Lambda(x)$ and then we can use the first property above.
- If we start with an odd sphere S^n, its rational cohomology is $H^*(S^n;\mathbb{Q}) \cong E(x)$ with $|x| = n$. In this case, we again obtain $S(n)$.
- If we start with an even sphere S^n, its rational cohomology is the same as above. In this case, though, $S(n)$ is not correct since the free symmetric algebra on an even-degree generator is polynomial. Therefore we must have $\Lambda[a, b]$ with $|a| = n, |b| = 2n - 1$. In order to kill off the square of a, we define the differential $d(a) = 0$ and $d(b) = a^2$. As a corollary, we see that $\pi_*(S^n) \otimes \mathbb{Q} = E(x_n, x_{2n-1})$ by determining the indecomposables of $\Lambda[a, b]$.
- If we start with $\mathbb{CP}^n$, we have $H^*(\mathbb{CP}^n;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^{n+1})$ with $|x| = 2$. Following the above line of argument, this must correspond to $\Lambda[a, b]$ with $|a| = 2, |b| = 2n + 1$ with $d(a) = 0$ and $d(b) = a^{n+1}$. As above, we see that $\pi_*(\mathbb{CP}^n) \otimes \mathbb{Q} \cong E(x_2, x_{2n+1})$.
- Note we can't repeat this for $\mathbb{RP}^n$ since it is not simply connected
- If we start with BU, we have $H^*(BU;\mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \ldots]$ with the Chern classes c_i having degree $|c_i| = 2i$. This then maps to $\Lambda[c_1, c_2, \ldots]$ with zero differential, and then we see that the rational homotopy groups are a copy of $\mathbb{Q}$ in every even degree.

Definition. If $A \in CDGA$ (or $X \in Top_{\mathbb{Q}}^{\geq 1, fin}$), then it is called *formal* if it has a (not necessarily Sullivan) model with $d = 0$.

Example. BU and S^n and $K(\mathbb{Q}, n)$ are formal. More generally, spheres, H-spaces, and Kähler manifolds are formal.

Note also that formal spaces are closed under wedge, sums, direct products, and connected sum of manifolds.

Why is formality important? One reason is that if A is formal, then all of its higher order Massey products vanish. Note that the converse is false.

Recall the later half of our diagram of Quillen equivalences

$$DGA \rightleftarrows CDGC \rightleftarrows DGL.$$

Next time, we want to understand

$$\mathcal{L} : CDGC \rightleftarrows DGL : \mathcal{C}$$

as well as the compositions with the dual adjunction $DGA \rightleftarrows CDGC$, which is denoted

$$L : DGA \rightleftarrows DGL : C$$

under with $A_X \overset{L}{\mapsto} L_X$. We will see that

$$\pi_*(\Omega X) \otimes \mathbb{Q} \cong H_*(L_X).$$

Remark. Eckmann-Hilton/Koszul duality between homotopy groups and cohomology says that we have a duality between giving a space a CW structure and building it out of a Postnikov tower. Applying rational homotopy to the Postnikov tower gives a commutative differential graded algebra, and we'll see that L_X can be obtained from the CW structure.

We'll also see next time that the category of connected differential Lie algebras DGL^{conn} is a model category with weak equivalences quasi-isomorphisms, fibrations surjective morphisms, and cofibrations the morphisms satisfying the left lifting property with respect to acyclic fibrations.

We'll also see that the category of connected, cocommutative, coaugmented differential graded algebras $CDGC^{conn}$ is a model category, with weak equivalences quasi-isomorphisms, cofibrations are injective morphisms, and fibrations are the morphisms satisfying the right lifting property with respect to acyclic cofibrations.