Rational Homotopy Theory Seminar

Week 5: Minicourse Part V

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Recall we were considering the chain of Quillen equivalences between (up to decoration) Top, $sSet$, $CDGA$, $CDGC$, and DGL.

Today we are considering the Quillen equivalences

$$(-)^\vee : CDGA^{fin} \rightleftarrows CDGC^{fin} : (-)^\vee$$

$$\mathcal{L} : CDGC^{\geq 1, coaug} \rightleftarrows DGL^{conn} : \mathcal{C}$$

$$L_* : CDGA^{fin, \geq 1} \rightleftarrows DGL^{conn, fin} : C_*$$

We'll assume that k is a field of characteristic zero. Recall the following definitions:

Definition. A *differential graded Lie algebra*, or *chain Lie algebra*, $L \in DGL$ is a chain complex of k-vector spaces

$$\cdot \to L_2 \xrightarrow{d} L_1 \xrightarrow{d} L_0$$

equipped with a Lie bracket $[-,-] : L_p \otimes L_q \to L_{p+q}$ satisfying

1. Antisymmetry $[x, y] = (-1)^{|x||y|+1}[y, x]$
2. Leibniz $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$
3. Jacobi (see notes from a couple of weeks ago)

The morphisms between differential graded Lie algebras are maps defined levelwise which commute with d and $[-,-]$. We say that L is connected if $L = L_{\geq 1}$.

Definition. A *differential graded coalgebra* $C \in DGC$ is a chain complex of k-vector spaces

$$\cdots \to C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0$$

with comultiplication $\psi : C \to C \otimes C$ and augmentation $\epsilon : C \to k$ satisfying

1. $d \circ d = 0$
2. $(id_C \otimes \psi) \circ \psi = (\psi \otimes id_C) \circ \psi$
3. $(id_C \otimes \epsilon) \circ \psi = (\epsilon \circ id_C) \circ \psi = id_C$
4. $\psi \circ d = (d \otimes id_C + \tau \circ (d \otimes id_C) \circ \tau) \circ \psi$

Up to sign, the fourth rule says that $\psi(dc) = \sum c' \otimes dc'' + dc' \otimes c''$ where $\psi(c) = \sum c' \otimes c''$.

The morphisms between differential graded coalgebras are maps defined levelwise which commute with d and ψ. We say that C is *cocommutative* if $\tau \circ \psi = \psi$, *coaugmented* if it has a map $\eta : k \to C$, and *connected* if $C_0 = k$.

We now want to define the free Lie algebra and the cofree differential graded coalgebra. We will define adjunctions

$$\mathbb{L} : Ch \rightleftarrows DGL : forget$$

$$F : Ch \rightleftarrows DGC : forget.$$

The *free Lie algebra* on V is defined as follows. Begin by considering

$$T(V) = \bigoplus_{k \geq 0} T^k(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

with bracket

$$[x, y] = x \otimes y + (-1)^{|x||y|+1} y \otimes x.$$

Then $\mathbb{L}(V)$ is the sub-Lie algebra of $T(V)$ generated by V.

The *cofree differential graded coalgebra* is defined using the same $T(-)$ functor as above. We have a comultiplication

$$\Delta : T(V) \to T(V) \boxtimes T(V)$$

$$\Delta(v_1 \otimes \cdots \otimes v_k) = \sum_{j=0}^{k} (v_1 \otimes \cdots \otimes v_j) \boxtimes (v_{j+1} \otimes \cdots \otimes v_k)$$

and we can define $F(V) = T(V)$.

We are now ready to define the Quillen equivalence mentioned earlier. We have three adjunctions

$$\mathcal{L} : CDGC^{conn} \rightleftarrows DGL^{conn} : \mathcal{C}$$

$$\Omega : CDGC^{conn} \rightleftarrows CDGH^{conn} : B$$

$$P : CDGH^{conn} \rightleftarrows DGL^{conn} : U$$

where $CDGH$ is the category of commutative differential graded Hopf algebras. We have the following definitions:

- $\mathcal{L} = P \circ \Omega$
- $P(H) = \{x \in H : \psi(x) = \lambda \otimes 1 + 1 \otimes x\}$, with bracket $[x, y] = x \otimes y + (-1)^{|x|} y \otimes x$
- $U(L) = TL / \{x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]\}$
- $\mathcal{C} = B \circ U$

The first adjunction is a Quillen equivalence, the third is an equivalence, and therefore the second is a Quillen equivalence.

Define

$$\mathcal{L}(C) = (\mathbb{L}(s^{-1}C_{>0}), \delta = \delta_0 + \delta_1)$$

where

$$\delta_0(s^{-1}x) = -s^{-1}d_C(x)$$

$$\delta_1(s^{-1}x) = -\frac{1}{2}\sum_i (-1)^{|sx'_i|}[s^{-1}x'_i, s^{-1}x''_i]$$

if $\psi(x_i) = \sum_i x'_i \otimes x''_i$.

Define

$$\mathcal{C}(L) = (Sym(sL), d = d_0 + d_1)$$

where

$$d_0(sx_1 \wedge \cdots \wedge sx_k) = \sum_i (-1)^{n_i}(sx_1 \wedge \cdots sd_C(x_i) \wedge \cdots sx_k)$$

$$d_1(sx_1 \wedge \cdots \wedge sx_k) = \sum_{i<j} (-1)^{|sk_i|+n_{ij}} s[x_k, x_j] \wedge sx_1 \wedge \cdots \wedge \widehat{sx_i} \wedge \cdots \wedge \widehat{sx_j} \wedge \cdots \wedge sx_k$$

where $sx_1 \wedge \cdots \wedge sx_k \in Sym(V)$ and $n_i = \sum_{j<i} |sx_j|$. This is also called the *Chevalley-Eilenberg construction.*

A *Lie model*, or a *Quillen model*, for a simply connected topological space $X \in Top^{\geq 1}$ is a differential graded Lie algebra $L_X \in DGL$ such that $C_*(L) \simeq \Omega^*_{poly}(S_\bullet(X))$.

Fact.

$$\pi_*(\Omega X) \otimes \mathbb{Q} \cong H_*(L_X)$$

Examples.

- Let $n \geq 2$. Then $L_{S^n} = (\mathbb{L}(\alpha), \delta(\alpha) = 0)$ where $|\alpha| = n - 1$.
- We can add in generators via $(L[\xi], \delta\xi = \alpha) = L * \mathbb{L}(\xi) = \mathbb{L}(L \oplus \mathbb{L}(\xi))/([x, y]_L = [x, y]_{\mathbb{L}}; [x, y]_{\mathbb{L}(\xi)} = [x, y]_{\mathbb{L}})$

Theorem. Consider the pushout $X \cup_f D^{n+1} = \text{colim}(D^{n+1} \leftarrow S^n \xrightarrow{f} X)$ where $\alpha \in L_{n-1}$ represents f. Let $L = L_X$. Then $(L(\xi), d\xi = \alpha)$ is a LIe model for $X \cup_f D^{n+1}$.

Example. Consider $\mathbb{CP}^2 = colim(D^4 \leftarrow S^3 \xrightarrow{\eta} S^2)$ where *eta* is one of the Hopf elements. Then $2\eta = [\iota_2, \iota_2] \in \pi_3(S^2)$ and $L_{\mathbb{CP}^2} = (\mathbb{L}(\iota, \xi), \delta\iota = 0, \delta\xi = \frac{1}{2}[\iota, \iota])$.