Rational Homotopy Theory Seminar

Week 6: Functor calculus and the rational Taylor tower of the identity

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Setup. Let $\mathcal{C}, \mathcal{D}$ be simplicial model categories, and suppose $F : \mathcal{C} \to \mathcal{D}$ is a finitary (commute with finite colimits), homotopy (preserves weak equivalences), continuous (i.e. $\mathcal{C}(c,d) \to \mathcal{D}(Fc, Fd)$ is a map of simplicial sets) functor with $F(*) \simeq *$.

We say that F is *n-homogeneous* if

$$F(-) \simeq \Omega^\infty C^{h\Sigma_n}(\Sigma^\infty -)^{\wedge n}$$

where C is a Σ_n object in $Sp(\mathcal{D})$, i.e. a spectrum with a Σ_n-action. If $\mathcal{D}$ is a category of spectra of $\mathcal{C}$ is a category of spectra, then can remove appropriate Σ^∞ or Ω^∞. This is like a homogeneous polynomial of degree n.

A functor F is *0-excisive* if $F \simeq *$ and F is *n-excisive* if either F is $(n-1)$-excisive or there exists H n-homogeneous and G $(n-1)$-excisive and fiber sequence $H \to F \to G$. Being n-excisive is like being polynomial of degree at most n.

Remark: this is equivalent to the cartesian/cocartesian cube definition of functor calculus.

Theorem. [Goodwillie, others for different categories] If F is a functor as above, then there exists an initial (up to weak equivalence) n-excisive functor under F called P_nF, i.e. $F \to P_nF$, and these fit into a tower called the *Taylor tower*:

$$\cdots \to P_nF \to P_{n-1}F \to \cdots$$

Definition/Theorem. Define

$$D_nF = hofib(P_nF \to P_{n-1}F).$$

Then D_nF is n-homogeneous, i.e.

$$D_nF(X) = \Omega^\infty(\partial_n(F) \wedge_{h\Sigma_n} (\Sigma^\infty X)^{\wedge n}).$$

The spectrum $\partial_n(F)$ is called the *n-th derivative of F*; it is a Σ_n-object in $Sp(\mathcal{D})$. These are like the n-th derivative evaluated at a point.

Motivating examples:

- $P_1(id_{Top_*}) = \Omega^\infty\Sigma^\infty$ and $\partial_1(F) = \mathbb{S}$
- $\pi_*(P_2(id_{Top_*})(X))$ is Mahowald's metastable homotopy groups of X
- If we're interested in computing $\pi_*F(X)$, then we can use the *Goodwillie spectral sequence*
$$E^1_{*,n} = \pi_*D_n(F)(X) = \pi_*^s\partial_n(F) \wedge_{h\Sigma_n} X^{\wedge n} \Rightarrow \pi_*F(X)$$
Behrens used this to recover the Toda range of unstable homotopy groups of spheres

Problems:

- The Σ_n action on $\partial_n(F)$ is not necessarily nice
- The maps in the Taylor tower are not Ω^∞-maps, so to compute differentials, one needs to explicitly understand the maps $P_nF \to P_{n-1}F$

Definition. If $u : F \to G$ is a natural transformation of functors, we say it *satisfies* $\mathcal{O}_n(c, \kappa)$ if for all spaces X which are k-connected with $k > \kappa \geq 0$, then the map $F(X) \to G(X)$ is $(-c + (n+1) \cdot k)$-connected.

Theorem. [Goodwillie, others for other categories] If $u : F \to G$ satisfies $\mathcal{O}_n(c, \kappa)$ for some c, κ, then $P_nF \to P_nG$ is a weak equivalence.

How do we find the Taylor tower for some functor F? We guess a tower

$$\cdots \to G_n \to G_{n-1} \to \cdots$$

with compatible natural transformations $u_i : F \to G_i$ such that G_n is n-excisive and u_n satisfies $\mathcal{O}_n(c, \kappa)$ for some c, κ. The theorem then guarantees that $P_nF \to G_n$ is a weak equivalence.

Rational homotopy theory. [Reference: B. Walter] Recall that $Sp_{\mathbb{Q}} \cong Ch_{\mathbb{Q}}$. We can ask about

$$\Sigma^{\infty+1} : DGL_{\mathbb{Q}} \leftrightarrows Ch_{\mathbb{Q}} : \Omega^{\infty+1}$$

where the $+1$'s are because we want to think of these as categories of loop spaces. Note that the right-hand side is the category of *unbounded* chain complexes, as opposed to the category we worked with for spaces. These are defined by

$$\Sigma^{\infty+1}(L) = L/[L, L] = L^{ab}$$

$$\Omega^{\infty+1}(C) = (C_{\geq 0}, [-, -] = 0).$$

We would like to compute $\partial_*(id_{DGL_{\mathbb{Q}}^{\geq 0, fin}})$ and from this get the Taylor tower.

The cofibrant replacement in $DGL_{\mathbb{Q}}^{conn, fin}$ is defined by sending

$$L \mapsto (\mathbb{L}(V), d)$$

where $\mathbb{L}$ is the free graded Lie algebra.

Define $\Gamma^1(L) = L$ and then $\Gamma^n(L) = [L, \Gamma^{n-1}(L)]$. The *lower central series* is then defined by

$$B_n(L) = L/\Gamma^n(L) = (\mathbb{T}^{\leq n}(V) \wedge \mathbb{L}(V), d)$$

and we define

$$H_n(L) = hofib(B_n(L) \to B_{n-1}(L) = (\mathbb{T}^{=n}(V) \cap \mathbb{L}(V)).$$

We have an explicit description

$$H_n(L) = (Lie(n) \otimes_{h\Sigma_n} (L^{ab})^{\otimes n})$$

where $Lie(n)$ is the graded vector space generated by all words $[x_1, [\dots[\dots, x_n]\dots]$ satisfying the Jacobi identity and anticommutative in degree 0. Therefore H_n is n-homogeneous, and by induction, $B_n(L)$ is n-excisive. We have compatible maps $L \to B_n(L)$, and if L is k-connected, then $L \to B_n(L)$ is $(n+1)k$-connected. By Goodwillie's theorem, we then have

$$P_n(id_{DGL}) \simeq B_n(-)$$

$$D_n(id_{DGL}) \simeq H_n(-)$$
$$\partial_n(id_{DGL}) \simeq Lie(n).$$

Further, the Goodwillie spectral sequence for id_{DGL} is a spectral sequence of Lie algebras.

Theorem. [Walter] If $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is a Quillen equivalence where L, R preserve all weak equivalences and R is finitary, then pre- and post-composition with L and R preserves Taylor towers, i.e.

$$P_n(F \circ R) = P_n(F) \circ R.$$

In words, we can "move" Taylor towers between certain Quillen equivalent categories.

Example. We then have

$$\partial_n(id_{CDGC_{\mathbb{Q}}^{fin,\geq 1}}) \simeq S^{1-n} \otimes Lie(n).$$

What happens if you try to lift this to the non-rational case? Then the derivative is called the *shifted Lie operad*:

$$\partial_*(id_{Top_*}) \simeq sLie$$

One way of defining $sLie$ is as the Koszul dual of the commutative operad $K(Comm)$.