M. Behrens: Koszul duality.

$\mathcal{C}, \otimes$ symm mon cat w. some notion of homotopy. Either $(Sp, \wedge)$ or $(Ch_k, \otimes)$ k comm ring.

$Seq(\mathcal{C})$ category of symmetric seq in $\mathcal{C}$. $Seq(\mathcal{C}) = \{A_* = \{A_i\}_{i\in\mathbb{N}} \mid \Sigma_i \curvearrowright A_i,\ A_i \in \mathcal{C}\}$

$A_* \in Seq(\mathcal{C}) \rightsquigarrow \mathcal{F}_{A_*}: \mathcal{C} \to \mathcal{C}$ $\mathcal{F}_A(X) = \bigvee_i A_i \wedge_{\Sigma_i} X^{\wedge i}$

$\circ$ is a monoidal structure (non symmetric), on $Seq(\mathcal{C})$. $(A_* \circ B_*)_n = \bigvee_{\substack{\text{partition} \\ n = n_1 + \dots + n_k}} (\Sigma_n)_+ \wedge_{\Sigma_{(n_1,\dots,n_k)}} A_k \wedge B_{n_1} \wedge \dots \wedge B_{n_k}$

k varies   subgroup of Σ_n preserving $n_1 + \dots + n_k$

$\mathcal{F}_A \circ \mathcal{F}_B = \mathcal{F}_{A\circ B}$. $1_* = (*, 1_{\mathcal{C}}, *, *, \dots)$ is the unit for $\circ$. $\mathcal{F}_{1_*} = id_{\mathcal{C}}$.

A monoid in $(Seq\,\mathcal{C}, \circ)$ is an operad in $\mathcal{C}$.

$\mathcal{O}$ operad then $\mathcal{F}_{\mathcal{O}} \circ \mathcal{F}_{\mathcal{O}} = \mathcal{F}_{\mathcal{O}\circ\mathcal{O}} \to \mathcal{F}_{\mathcal{O}}$ so $\mathcal{F}_{\mathcal{O}}$ is a monad.

A $\mathcal{O}$-Alg in $\mathcal{C}$ is an alg over $\mathcal{F}_{\mathcal{O}}$. So $\mathcal{F}_{\mathcal{O}}(A) \to A \rightsquigarrow \bigvee_i \mathcal{O}_i \wedge_{\Sigma_i} A^{\wedge i} \to A$

Coming from unit map

An operad $\mathcal{O}$ is reduced if $\mathcal{O}_0 = *$ and $1_{\mathcal{C}} \to \mathcal{O}_1$ is an iso.

Ex: $Comm_*$ = reduced comm operad. $Comm_*^{un}$ ← unreduced $= (1, 1, 1, \dots)$. $Comm_* = (*, 1, 1, 1, \dots)$. Alg's / $Comm_*^{un}$ are comm algs in $\mathcal{C}$ w. unit coming from $Comm_0^{un} = 1$, so Algs / $Comm_*$ are non-unital comm algs, which are equiv to augmented unital algs in $\mathcal{C}$. (think H^* vs. $\tilde{H}^*$)

add Top if in Sp

André-Quillen ~~coh~~ homology / Cotangent complex.

$1_* \to \mathcal{O} \to 1_* \rightsquigarrow \mathcal{F}_{\mathcal{O}} \to id$ so $id \circ \mathcal{F}_{\mathcal{O}}$

indecomposables

$\mathcal{O}$ reduced operad. $A \in Alg_{\mathcal{O}}(\mathcal{C}) \rightsquigarrow QA \in \mathcal{C}$ $Q(A) = Coeq\left(id \circ \mathcal{F}_{\mathcal{O}}(A) \rightrightarrows \mathcal{F}_{\mathcal{O}}(A)\right)$

$\mathcal{F}_{\mathcal{O}}(A) \to A$

A augmented alg, IA aug-ideal, $IA/IA^2 = QA = \Omega_I A$

Left derived

$\mathbb{L}Q(A) \simeq B(id, \mathcal{F}_{\mathcal{O}}, A) = \left| id(A) \leftleftarrows id \circ \mathcal{F}_{\mathcal{O}}(A) \leftleftarrows \cdots \right|$

(Top) André Quillen hom

Descent: (After Hess-Lurie-Arone Ching):

Ex: $Top_* \underset{\Sigma^\infty}{\overset{\Omega^\infty}{\rightleftarrows}} Sp$. $\Sigma^\infty X$ is a coalg / $\underbrace{\Sigma^\infty \Omega^\infty}_{=:K}$. $HoTop_* \xrightarrow{\Sigma^\infty} Ho(~~comod~~\ Alg_K(Sp))$ can X be recovered from $\Sigma^\infty X$ + K-structure, is the functor faithful or full? Mostly no!

$\mathrm{Ho}(\cancel{\mathrm{Comod}}\,\mathrm{Alg}_K(Sp)) \longrightarrow \mathrm{Ho}\,\mathrm{Top}_*$ $\quad C \longmapsto C(\Omega^\infty, \Sigma^\infty\Omega^\infty, C)$. $\quad C(\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty X) = \mathrm{Tot}\big(\Omega^\infty\Sigma^\infty X \rightrightarrows (\Omega^\infty\Sigma^\infty)^2 X \rightrightarrows \cdots\big)$

This is an equiv if X is simply connected. So X simply conn htpy type can be recovered from $\Sigma^\infty X$ + decent data (K-data)

Koszul Duality: $Q : \mathrm{Alg}_{\mathcal{O}} \rightleftarrows \mathcal{C} : \mathrm{triv}$, $Z \in \mathcal{C}$, $\mathcal{O}_n \wedge_{\Sigma_n} \mathrm{triv}\, Z^{\wedge i} \xrightarrow{0} \mathrm{triv}\, Z$ $\quad n > 1$ $\mathcal{O}_1 = \mathbb{1}$ $\quad \mathcal{O}_1 \wedge \mathrm{triv}\, Z \cong \mathrm{triv}\, Z$

$\mathbb{L}Q : \mathrm{Ho}\,\mathrm{Alg}_{\mathcal{O}}\mathcal{C} \rightleftarrows \mathrm{Ho}\,\mathcal{C} : \mathrm{triv}$ $\quad K = \mathbb{L}Q \circ \mathrm{triv}$ comonad. $K(Z) = B(\mathrm{id}, \mathcal{F}_{\mathcal{O}}, \mathrm{triv}\, Z) \simeq B(\mathrm{id}, \mathcal{F}_{\mathcal{O}}, \mathrm{id})(Z) = \mathcal{F}_{B\mathcal{O}}(Z)$

So what is $B\mathcal{O}$? $\mathcal{O}$ monoid in $(\mathrm{Seq}(\mathcal{C}), \circ)$ so form bar construction $B\mathcal{O} = B(1_*, \mathcal{O}_*, 1_*) = |\, 1_* \leftleftarrows \mathcal{O}_* \cdots \,|$

Claim: $B\mathcal{O}$ cooperad $\Rightarrow \mathcal{F}_{B\mathcal{O}}$ is a comonad (Ginzburg-Kapranov, Getzler-Jones, Fresse (Ch_k), Ching (Sp))

Livernet, Francis-Gaitsgory: $B\mathcal{O} = 1 \circ^L_{\mathcal{O}} 1 \simeq 1 \circ^L_{\mathcal{O}} \mathcal{O} \circ^L_{\mathcal{O}} 1 \xrightarrow{\text{Augmentation}} 1 \circ^L_{\mathcal{O}} 1 \circ^L_{\mathcal{O}} 1 \simeq 1 \circ^L_{\mathcal{O}} 1 \circ 1 \circ^L_{\mathcal{O}} 1 = B\mathcal{O} \circ B\mathcal{O}$

$\mathcal{F}_{B\mathcal{O}}$-CoAlg:
$$\begin{tikzcd} A \arrow[r, "\text{Ind-coalg}"] \arrow[dr, "B\mathcal{O}\text{-coalg}"'] & \bigvee B\mathcal{O}_i \wedge_{\Sigma_i} A^{\wedge i} \arrow[d, "N"] \\ & \prod B\mathcal{O}_i \wedge^{\Sigma_i} A^{\wedge i} \end{tikzcd}$$
This structure is called a dp-$B\mathcal{O}$-Coalg

So $\mathrm{Alg}_{\mathcal{O}} \xrightarrow{\mathbb{L}Q} \mathrm{CoAlg}^{dp}_{B\mathcal{O}}$ analyse faithfulness etc..

$B\mathrm{Lie}_* = s\,\mathrm{Comm}^\vee$ and $Bs^{-1}\mathrm{Lie} = \mathrm{Comm}^\vee$.

So $\mathbb{Q}$-htpy: $\mathrm{Ho}(\mathrm{Alg}^{\geq 1}_{\mathrm{Lie}}(Ch_{\mathbb{Q}})) \simeq \mathrm{Ho}(\mathrm{Coalg}^{\geq 1}_{s\mathrm{Comm}}(Ch_{\mathbb{Q}})) \simeq \mathrm{Ho}(\mathrm{Coalg}^{\geq 2}_{\mathrm{Comm}}(Ch_{\mathbb{Q}}))$

$\mathrm{Ho}(Sp_{\mathbb{Q}}) \simeq \mathrm{Ho}(Ch_{\mathbb{Q}})$

$\Sigma^\infty\Omega^\infty \simeq \mathcal{F}_{\mathrm{Comm}^\vee}$