Rational Homotopy Theory Seminar

Week 9: Examples and criteria for formality

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Geometric motivation. Suppose given two links $L_1, L_2 \in S^3$.

[picture of links]

From this data, one can define the *linking number* $Link(L_1, L_2)$. Take $L_1 : S^1 \to S^3$ and fill it in along the inclusion $S^1 \hookrightarrow D^2$ to get a map $D_1 : D^2 \to S^3$. Then we have

$$Link(L_1, L_2) = \sum_{inter, L_2, D_1} \{\pm 1\}.$$

Gauss showed

$$Link(L_1, L_2) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{det(\dot{L_1}, \dot{L_2}, L_1 - L_2)}{|L_1 - L_2|^3}.$$

Recall. Given a map $f : S^3 \to S^2$, how to determine if it's null-homotopic? Choose two points $x_0, x_1 \in S^2$ and consider their (generically one-dimensional curves in S^3) fibers $f^{-1}(x_i)$. The *Hopf invariant* is defined by

$$H(f) = Link(f^{-1}(x_0), f^{-1}(x_1)) \in \mathbb{Z}.$$

How do we see that this is homotopy invariant? We'll define a new Hopf invariant which is clearly homotopy invariant, and then we'll show that it agrees with the old definition.

First, take a 2-form $\omega \in \Omega^2(S^2)$ with $\int \omega = 1$, e.g. some normalized volume form or a bump function. Pulling back gives $f^*\omega \in \Omega^2(S^3)$, but since $H^2(S^3) = 0$, we must have some $\eta \in \Omega^1(S^3)$ such that $f^*\omega = d\eta$. Then we define the new Hopf invariant by

$$\text{“}H(f)\text{”} = \int_{S^3} f^*\omega \wedge \eta.$$

Exercise. Homotoping f, ω, η doesn't change "$H(f)$". Hint: use Stokes theorem.

How do we relate this construction to the linking number definition? By the exercise, we can choose f to be transverse, ω to be a bump function around some point we're interested in, say x_1, and then when you look at the fiber, you get a tubular neighborhood η around $f^{-1}(x_1)$.

[picture of tubular neighborhood around $f^{-1}(x_1)$ with D_0 the image of the disk with transverse intersection]

[picture of Borromean rings]

The linking number of any two rings is zero, but they cannot be unlinked.

Remark. Mathematical interpretation of the Trinity using Borromean rings (trinitas/unitas).

Exercises. Define $B = S^3 \setminus K$ where K is the Borromean rings.

1. Compute H^*B as a group

2. Find a minimal model for B. Interpret the products in the minimal model using cup product.

3. Interpret these elements in terms of God

Reference. Deligne-Sullivan-Morgan.

Recall. Let $\mathcal{M}$ be a minimal commutative differential graded algebra. Suppose we have a filtration

$$K \hookrightarrow \mathcal{M}_1 \hookrightarrow \mathcal{M}_2 \hookrightarrow \cdots \hookrightarrow \bigcup \mathcal{M}_i = \mathcal{M}$$

such that as an algebra,

$$\mathcal{M}_{i+1} \cong \mathcal{M}_i \otimes SymV$$

and such that $d_{\mathcal{M}_{i+1}}|_{\mathcal{M}_i} = d|_{\mathcal{M}_i}$ and $d_{\mathcal{M}_i}(V) \subset \mathcal{M}_i$.

Massey products. Suppose we're given three cohomology classes $[x], [y], [z] \in H^*(\mathcal{M})$ such that $[x] \cup [y] = 0$ and $[y] \cup [z] = 0$. By associativity, we have

$$[x] \cup [y] \cup [z] = 0,$$

and it's zero "for two reasons".

Returning to the minimal model, suppose we have $x, y, z \in \mathcal{M}$. Then if $x \wedge y = ds$ and $y \wedge z = dt$, then we can define the *Massey product* by

$$\langle x, y, z \rangle = [s \wedge z + (-1)^{|x|} x \wedge t] \in H^*(\mathcal{M}).$$

Note that we made choices of s and t above which might affect the Massey product.

Exercise. Varying choices above changes $\langle x, y, z \rangle$ by an element in $\{xH^*(\mathcal{M}) + zH * (\mathcal{M})\}$.

Examples.

- Gluing disks along the equator for S^2.

- An analogous construction using bordism groups.

Example. Consider the minimal model which is k in degree 0, 0 in degree 1, x, y in degree 2 with $d(x) = 0$ and $d(y) = 0$, u, v in degree 3 which have $d(u) = x^2$ and $d(v) = xy$. Then we have

$$\langle x, x, y \rangle = [u \wedge y - x \wedge v].$$

We need to determine the indeterminacy subgroup of this Massey product, since if the class lives there it's zero. The indeterminacy group is

$$\{xH^3(\mathcal{M}) + yH^3(\mathcal{M})\} = \{0\}.$$

Definition. Let $\mathcal{M}$ be a minimal model. Then $\mathcal{M}$ is *formal* if there exists a map

$$f : \mathcal{M} \to H^*\mathcal{M}$$

such that f is an algebra homomorphism and f is a quasi-isomorphism.

Example. If $\mathcal{M}$ is formal and all Massey products have no indeterminacy, then all Massey products must be trivial. To see this, note that you're formal if and only if any you're isomorphic to a cdga with zero differential. Since Massey products are invariants of quasi-isomorphism class, all Massey products are zero.

Theorem. Suppose $\mathcal{M}$ is generated by $\bigoplus V^i$ where $C^i \subset V^i$ are the closed elements. Then $\mathcal{M}$ is formal if and only if there exists a splitting $\pi : V^i \to C^i$ (i.e. $V^i \simeq N^i \oplus C^i$) such that if $\alpha \in I(\bigoplus N^i)$ with $d\alpha = 0$, then α is exact.

In particular, formality is stronger than just all Massey products vanishing.