Operads and Loop Space Machinery Seminar
Week 1: Minicourse Part I, Loop spaces and spectra
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1. Overview of minicourse

1.1. **References.** The minicourse will be based on the following texts:

(1) Adams, "Infinite loop spaces"
(2) May, "The geometry of iterated loop spaces"
(3) Cohen-Lada-May, "The homology of iterated loop spaces"
(4) Hatcher, "Algebraic topology"

1.2. **Outline.** An outline for the minicourse:

(1) Motivation for loopspace theory. Generalized cohomology theories, Brown representability theorem, spectra, additional structure. Overview of applications and some possible topics for talks after the minicourse.
(2) Operads. Definition and examples. Some technical lemmas.
(3) The approximation theorem. Key ingredients and sketch of proof.
(4) The recognition principle. Key ingredients and sketch of proof. Homology of iterated loop spaces.

2. Cohomology theories

The primary tools for understanding spaces in algebraic topology are *cohomology theories*. We begin by recalling their definition.

Definition 2.1. A *cohomology theory* E is a sequence of contravariant functors $E^n : (X,A) \mapsto E^n(X,A)$, $n \in \mathbb{Z}$, from the category of pairs of topological spaces to the category of abelian groups, together with a natural transformation $d : E^i(X,A) \to E^{i+1}(A)$. These are required to satisfy the *Eilenberg-Steenrod axioms*:

(1) (homotopy) If $f, g : (X,A) \to (Y,B)$ are homotopic maps, then the induced maps $f^*, g^* : E^n(Y,B) \to E^n(X,A)$ are the same for all $n \in \mathbb{Z}$.
(2) (excision) If $X = A \cup B$, then the inclusion $f : (A, A\cap B) \to (X,B)$ induces an isomorphism $E^n(X,B) \xrightarrow{f^*} E^n(A, A\cap B)$ for all $n \in \mathbb{Z}$
(3) (dimension) $E^n(pt) = 0$ for all $n \neq 0$.
(4) (additivity) If $X = \coprod_\alpha X_\alpha$ is a disjoint union of spaces, then $E^n(X) \cong \prod_\alpha E^n(X_\alpha)$ for all $n \in \mathbb{Z}$
(5) (exactness) Each pair (X,A) induces a long exact sequence in cohomology via the inclusions $f : A \to X$ and $g : (X, \emptyset) \to (X,A)$,

$$\cdots \to E^i(X,A) \to E^i(X) \to E^i(A) \to E^{i+1}(X,A) \to \cdots .$$

We say that E is a *generalized cohomology theory* if it satisfies all of the Eilenberg-Steenrod axioms except possibly dimension.

Remark 2.2. *We can similarly define (generalized) homology theories*

Example 2.3. *We list a few examples here:*

(1) *Singular cohomology with coefficients in an abelian group A, denoted* $H^*(-;A)$
(2) *de Rham cohomology, denoted* $H^*_{dR}(-)$
(3) *Topological K-theory, denoted* $KO^*(-)$ *or* $KU^*(-)$
(4) *Cobordism, denoted* $MO^*(-)$ *or* $MU^*(-)$

3. From cohomology theories to Ω-spectra

Definition 3.1. We will denote homotopy classes of pointed maps between pointed spaces X and Y by

$$[X, Y] = \{f : X \to Y\}/(f \sim g \text{ if } f \text{ and } g \text{ are homotopic}).$$

The *n-th homotopy group* of a space X is then

$$\pi_n(X) := [S^n, X]$$

where S^n is the n-sphere.

We'll examine the first example in more detail here. A definition of singular cohomology with coefficients in $\mathbb{Z}$ can be found in Section 2.1 of Hatcher's "Algebraic Topology". We'll give a different but equivalent definition here for any abelian group A.

Let $K(A, n)$ denote the Eilenberg-MacLane space for A. This is a space characterized by the property that

$$\pi_i(K(A, n)) = \begin{cases} A & i = n, \\ 0 & \textit{else}. \end{cases}$$

Writing A in terms of generators and relations, the construction of this space as a CW complex can roughly be described as realizing this presentation using n and $n+1$-spheres, then attaching larger dimensional cells to kill higher homotopy groups.

The n-th singular cohomology of X with coefficients in A can be defined as

$$H^n(X; A) := [X, K(A, n)].$$

In the language of category theory, the space $K(A, n)$ *represents* the functor H^n in the category of based spaces and homotopy classes of maps.

Recall that there is an adjunction

$$\Sigma : Top_* \leftrightarrows Top_* : \Omega$$

where Σ denotes reduced suspension,

$$\Sigma X = (S^1 \times X)/(S^1 \vee X) \cong S^1 \wedge X,$$

and Ω denotes (based) loops,

$$\Omega X = Maps_*(S^1, X).$$

Then we have

$$\pi_i(K(A, n)) = [S^i, K(A, n)] \cong [\Sigma S^{i-1}, K(A, n)] \cong [S^{i-1}, \Omega K(A, n)]$$

from which we conclude that $K(A, n) \cong \Omega K(A, n+1)$.

In other words, to define $H^n(-;A)$ for all $n \in \mathbb{Z}$, we just needed to define $K(A,0)$ and then understand the "deloopings" of $K(A,0)$, i.e. spaces X which are models for $K(A,n)$ in that $\Omega^n X \cong K(A,0)$. This seems like a silly thing to think about since we've already seen how to construct $K(A,n)$ for all $n \geq 0$, but we'll see below that it's a useful idea. First we need a definition.

Definition 3.2. An Ω-spectrum is a sequence of based spaces $\{K_n\}$, $n \geq 0$, together with homeomorphisms $K_n \xrightarrow{\cong} \Omega K_{n+1}$.

Example 3.3. *We have just shown that* $\{K(A,n)\}$, $n \geq 0$, *form an* Ω*-spectrum.*

Theorem 3.4. *(Brown representability theorem) Every reduced cohomology theory has the form*

$$E^n(X) = [X, K_n]$$

for some Ω*-spectrum* $\{K_n\}$.

Proof. See Section 4.E of Hatcher's "Algebraic Topology". □

We can also verify that every Ω-spectrum gives rise to a cohomology theory. Therefore we have a correspondence between cohomology theories and Ω-spectra.

4. From Ω-spectra to infinite loop spaces

By definition, the 0-th space in an Ω-spectrum is homeomorphic to the iterated loop spaces of the i-th spaces, i.e.

$$K_0 \cong \Omega K_1 \cong \Omega^2 K_2 \cong \cdots \cong \Omega^n K_n \cong \Omega^{n+1} K_{n+1} \cong \cdots.$$

Definition 4.1. We say that X is an *n-fold loop space* if X satisfies $X \cong \Omega^n Y$ for some space Y. We say that X is an *infinite loop space* if $X = \Omega^n Y_n$ for some spaces Y_n and all $n \geq 1$.

We can define a functor

$$\Omega^\infty : \Omega - Sp \to Top_*$$

by the assignment $E \mapsto E_0$. If X is an infinite loop space, we can define a functor the opposite direction

$$\Sigma^\infty : Top_* \to Sp$$

by setting $(\Sigma^\infty X)_n := \Sigma^n X$.

By the previous discussion, E_0 is an infinite loop space. Therefore we have the following correspondences:

$$\text{(cohomology theories)} \overset{(1)}{\leftrightarrow} (\Omega\text{-spectra}) \overset{(2)}{\leftrightarrow} \text{(infinite loop spaces)}$$

Here (1) is given by the Brown representability theorem and (2) is Ω^∞ and Σ^∞.

Example 4.2. *At this point, we mention another example of a cohomology theory/*Ω*-spectrum/infinite loop spaces where this story is fully understood. If* X *is a nice space, we can define the complex topological K-theory of* X *by*

$$K^0(X) := Gr(Vect(X)).$$

Here, $Gr(-)$ is the Grothendieck group completion of a commutative monoid, and $Vect(X)$ is the monoid of isomorphism classes of complex vector bundles over X with monoidal structure given by direct sum. Then one can verify that

$$K^0(X) \cong [X, BU]$$

where BU is the classifying space of the unitary group.

Complex Bott Periodicity shows that we have

$$\Omega U \simeq BU \times \mathbb{Z},$$

$$\Omega(BU \times \mathbb{Z}) \simeq U.$$

Since $\Omega BU \simeq \Omega(BU \times \mathbb{Z})$, we see that BU is an infinite loop space. Therefore $K^0(-)$ extends to a cohomology theory $K^n(-)$.

Bott Periodicity is a stunning example of understanding iterated loops of a space. In general, it's very difficult to understand $\Omega^n X$ for any n or X.

5. Additional structure in cohomology theories

There is some additional structure in the cohomology theories we mentioned above.

Example 5.1. *All of the cohomology theories described above are actually* multiplicative cohomology theories. *We briefly mention the source of these additional structures here:*

(1) *In singular cohomology with coefficients in a* ring R, *we can define a* cup product pairing

$$H^i(X; R) \times H^j(X; R) \to H^{i+j}(X; R).$$

See, for example, Section 3.2 of Hatcher's "Algebraic Topology". This additional structure for $R = \mathbb{F}_2$ can be used to distinguish between spaces like $\mathbb{R}P^n$ and $\bigvee_{i=1}^n S^i$.

(2) *In de Rham cohomology, wedge product of differential forms gives rise to a similar multiplicative structure in $H^*_{dR}(M; \mathbb{R})$.*

(3) *In topological K-theory, tensor product of vector bundles gives rise to a similar multiplicative structure in $K^*(X)$*

(4) *In complex cobordism, Cartesian product of manifolds gives rise to a similar multiplicative structure in $MU^*(X)$*

Note that many of these products correspond to *geometric* constructions. In other words, we might suspect that these structures are easier to understand or produce in the infinite loop space corresponding to a cohomology theory.

There is even more structure in some of these cohomology theories, called *cohomology operations.*

Example 5.2. *In the case of mod p singular cohomology, $H^*(X; \mathbb{F}_p)$ is a module over the* Steenrod algebra $\mathcal{A}$. *When $p = 2$, one can show that*

$$\mathcal{A} \cong \langle Sq^i | i \geq 0 \rangle / \sim$$

where $\sim$ *is the* Adem relations*. Here,*

$$Sq^i : H^n(X;\mathbb{F}_2) \to H^{n+i}(X;\mathbb{F}_2).$$

Cohomology operations can be difficult to understand at the level of infinite loop spaces; we will address a dual notion, *homology operations*. In the case of mod p singular homology, these are called *Araki-Kudo-Dyer-Lashof operations* and have been used extensively in many areas of stable homotopy theory.

6. Goals for the minicourse

The goal of the minicourse is to address the following questions:

(1) How do we know if a topological space X is an (infinite) loop space? This is the recognition principle which we will work towards. In particular, we will see that X is an n-fold loop space if it receives an action of a certain operad (to be defined next week).
(2) What can we say about the homotopy type of an infinite loop space? For Eilenberg-MacLane spaces this was by definition, and for BU and BO we needed Bott Periodicity. More generally, we will see that the construction above gives techniques for understanding the homology of such spaces.
(3) What structure can be understood in cohomology by understanding structure in the corresponding infinite loop space? This is closely related to the previous question.