Talk outline

Monday, November 6, 2017 11:55 AM

Intro

Goal: Give an account of $H_*(D_d(n)) = H_*(D_d(n); \mathbb{Z})$ ← little d-disks
which form an operad that acts on $H_*(\Omega^d X)$
via nice operations

follow Sinha, much of original is Cohen

Recall $D_d(n) = \{$framed emb. of ordered n copies of D^d into D^d disjoint$\}$ ← open disks

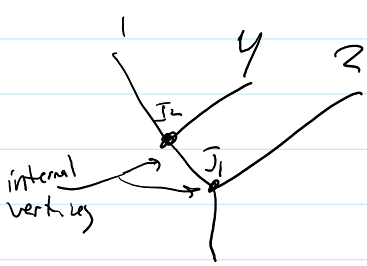
Let $F_n(X) = \{(x_1, \dots, x_n) \in X^n$ distinct$\}$.

Then
$$D_d(n) \xrightarrow{\text{centers}} F_n(D^d) \xrightarrow{\cong} F_n(\mathbb{R}^d)$$
[inverse htpy eq: config → use radius = 1/2 the min that would work]

is a htpy eq

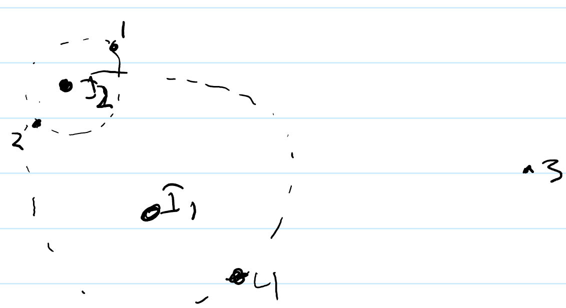
Recall $F_2(\mathbb{R}^d) \simeq S^{d-1}$ [shift pt 1 to origin & scale]
Want to generalize to F_n

Given a rooted binary tree T w/ left & right distinguished
leaf labels $\subseteq \{1, \dots, n\}$



let $|T|$ = # internal vertices
[= # leaves − 1]

Define a map $P_T : (S^{d-1})^{|T|} \to F_n(\mathbb{R}^d)$ [pt → where left parent goes]



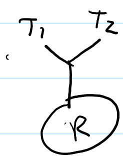
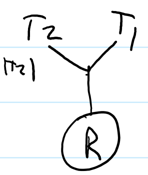
(distances scale by $\varepsilon < 1/3$ away from root)

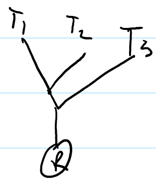
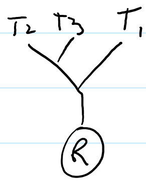
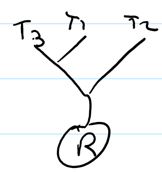
Let $p_T \in H_{|T|(d-1)}(f_n(\mathbb{R}^d))$ image of fund. class

Extend to forests with leaves $= \{1, \dots, n\}$

<u>Def</u> Let forest_n = free module on these forests
then have map
$$\varphi \; \text{forest}_n \longrightarrow H_*(f_n(\mathbb{R}^d))$$

<u>Thm</u>: $\operatorname{Ker} \varphi$ contains:

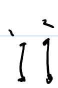
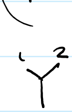
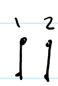
(1)  $+(-1)^{(d-1)|T_1||T_2|}$ 

(2)  $+$  $+$ 

(3) Permuting trees in forest f by σ : $\sigma f = (\operatorname{sgn} \sigma)^{d-1} f$

<u>Pf</u> (1) Antipodal map on one factor
(3) permutation on $(S^{d-1})^{|F|}$
(2)
idea: define a submanifold where the centers of T_i vary

Let $\mathrm{Pois}^d(n) = \text{forest}_n / (1)(2)(3)$. $\left(\mathrm{Pois}^d(n) \longrightarrow H_*(f_n(\mathbb{R}^d))\right)$

Pois^d is "generated" by  and  s.t.  $\to H_0(f_n(\mathbb{R}^d))$
represents Pontryagin product and  in $H_{d-1}(f_n(\mathbb{R}^d))$
is some bracket-like op

$$\left[\; H_i(D_d(n)) \otimes H_{a_1}(X) \otimes \cdots \otimes H_{a_n}(X) \longrightarrow H_{i+a}(X) \; \right] \quad a = \Sigma a_j$$

WTS map

$$\mathrm{Pois}^d(n) \to H_*(f_n(\mathbb{R}^d)) \text{ is iso}$$

Strategy: Construct

$$\mathrm{Siop}^d(n) \to H^*(f_n(\mathbb{R}^d))$$

& perfect pairing on $\mathrm{Pois}^d \times \mathrm{Siop}^d$ which agrees w/ homology-cohomology pairing $\Rightarrow$ maps are inj.

Use fibrations to upper bound dim $\Rightarrow$ surj

Cohomology:

[Define maps $f_n(\mathbb{R}^d) \to$ spheres by "unit vector" from $x_i \to x_j$]

Def Let $\alpha_{ij}: f_n(\mathbb{R}^d) \to S^{d-1}$ be

$$(x_1, \dots x_n) \mapsto \frac{x_j - x_i}{|x_j - x_i|}$$

Let

$$a_{ij} \in H^*(f_n(\mathbb{R}^d)) \text{ be } \alpha_{ij}^*(\iota)$$

[Extend this to graphs]

Def Let $\Gamma(n)$ = free module on directed graphs on $\{1, \dots n\}$ with an order on edges [The order records the symmetric grp action] w/multiplication as union

Then a_{ij} extends to a map $\Gamma(n) \to H^*(f_n(\mathbb{R}^d))$

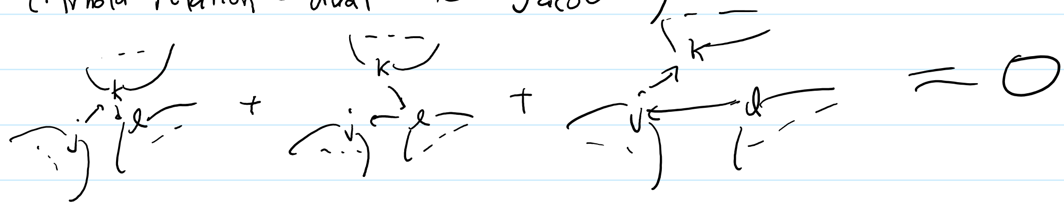
Thm $\mathrm{Ker}(\Gamma(n) \to H^*(f_n(\mathbb{R}^d)))$ contains

<u>Thm</u> $\operatorname{Ker}(\Gamma(n) \to H^*(f_n(\mathbb{R}^d)))$ contains

(1) $i \to j = (-1)^{d-1}\; j \to i$

(2) $i_1 \to j_1 \;\; i_2 \to j_2 = (-1)^d \;\; i_2 \to j_2 \;\; i_1 \to j_1$

(3) (Arnold relation ~ dual to Jacobi)



Pf: Omitted

<u>Def</u> Let $Siop^d(n) = \Gamma(n) / (1)+(2)+(3)$

$\Rightarrow$ Have map $Siop^d(n) \to H^*(f_n(\mathbb{R}^d))$

<u>Pairing</u>

Let $\langle -,- \rangle_H$ = homology-cohomology pairing

<u>Lemma</u>

$$\left\langle {}^{k}\curlyvee^{l},\; i \to j \right\rangle_H = \begin{cases} 1 & i=k,\ j=l \\ 0 & \text{else} \end{cases}$$

<u>Pf</u> Is equal to degree of

$$S^{d-1} \xrightarrow{k \curlyvee l} f_n(\mathbb{R}^d) \xrightarrow{i \to j} S^{d-1}$$

Extend to $Siop^d(n) \times Pois^d(n)$

Define

$$\langle G, f \rangle_c = \begin{cases} 0 & \text{if } i \to j \in G \ \&\ i,j \text{ in diff trees} \\ \pm 1 & \text{if } {}^{a}\curlyvee^{b} \Rightarrow a \to b \ \& \end{cases}$$

$\left(\pm 1 \text{ if } \overset{a\ b}{\vee} \Rightarrow a \nearrow b \right.$ &

$\overset{a\ b\ c}{\vee\!\vee} \Rightarrow$ exactly one of $a \nearrow c$ or $b \nearrow c$

etc

[Questions: give β def etc]

Thm $\langle G, f \rangle_C = \langle G, f \rangle_H$

Pf Consider map

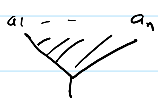
$$(S^{d-1})^{|F|} \longrightarrow f_n(\mathbb{R}^d) \longrightarrow (S^{d-1})^{|G|}$$

& Take homotopy as radius $\to 0$

Thm $\langle -, - \rangle_C$ is perfect [$\mathrm{Pois}^d \xrightarrow{\cong} (\mathrm{Siop}^d)^{op}$]

Pf sketch

- Pois^d is spanned by forests of trees



, $a_1 < a_i$ $i > 1$

- Siop^d spanned by graphs whose components are

$$a_1 \to a_2 \to \cdots \to a_n \qquad a_1 < a_i, \text{ ordered in order}$$

- Pairing of these is $\begin{cases} 1 & \text{ordered portions are same} \\ 0 & \text{else} \end{cases}$

- Thus basis, not just span,
  & maps are injective

Now, want to upper bound $\dim H_j(f_n(\mathbb{R}^d))$

Have fibration

$$\mathbb{R}^d - n\text{pts} \longrightarrow F_{n+1}(\mathbb{R}^d)$$

$$\begin{array}{ccc} \mathbb{R}^d - n\text{-pts} & \longrightarrow & F_{n+1}(\mathbb{R}^d) \\ \text{IS} & & \Big\downarrow \text{forget } n+1^{st} \\ \bigvee_{i=1}^{n} S^{d-1} & & F_n(\mathbb{R}^d) \end{array}$$

Use Serre Spectral Sequence for $f \to E \to B$; $\pi_1(B) = 0$ & $H_*(f)$ free

$\Rightarrow E^2 = E^\infty$ &

$$rk\left(H^i\, F_{n+1}(\mathbb{R}^d)\right) = rk\left(H^i(F_n(\mathbb{R}^d))\right) + rk\left(H^{i-d+1}(F_n(\mathbb{R}^d))\right) \cdot n$$

Let $P(i,n,d) = rk(H^i \quad)$

$$P(i,n+1,d) = P(i,n,d) + n \times P(i-d-1,n,d)$$

Let $Q(i,n,d) = rk\ Siop^d_i(n)$ (d-1) tram edges (?)

basis = long graphs
remaining edges argument

(Proved?)

<u>Thm</u> $Siop^d(n) \longrightarrow H^*(F_n\mathbb{R}^d)$ is an iso of graded $\mathbb{Z}$-mods

<u>Cor</u> $Pois^d(n) \longrightarrow H^*(F_n\mathbb{R}^d)$ is o

<u>Pf Cor</u> universal coeffs

$Pois^d$ is a graded operad generated by

$$\overset{1}{\bullet}\ \overset{2}{\bullet} \in Pois_0^d(2)$$

$$\overset{1}{\ }\!\!\curlyvee\!\!\overset{2}{\ } \in Pois_{d-1}^d(2)$$

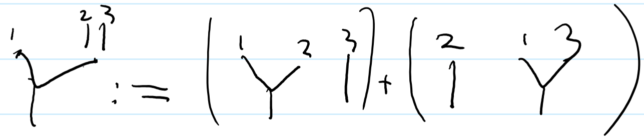
giving ops $\cdot : H_p(X) \otimes H_q(X) \longrightarrow H_{p+q}(X)$

$\lambda_{d-1} : H_p(X) \otimes H_q(X) \longrightarrow H_{p+q+d-1}(X)$

Have $\lambda_{d-1}(x,-)$ deriv

$$\lambda_{n-1}(x, yz) = \lambda_{n-1}(x, y)z + y\lambda_{n-1}(x, z)$$

Need to define



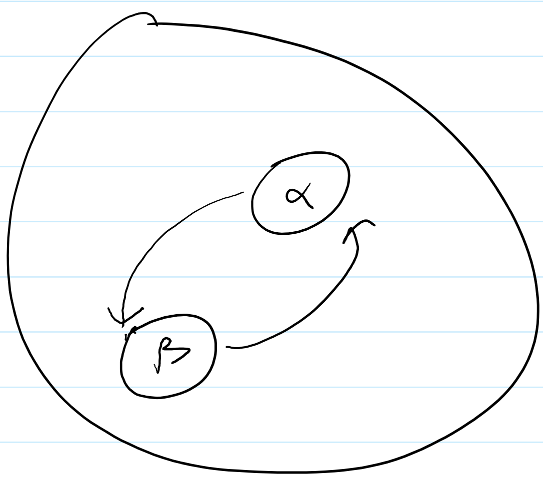
Then,

Thm $\mathrm{Pois}^d \longrightarrow H_*(F_\bullet \mathbb{R}^d)$ iso of operads
[$S_{iop}^d \longrightarrow H^*(F_\bullet \mathbb{R}^d)$ iso of co-operads]

Picture: λ_1 on $\Omega^2 X$

Given cycles $\alpha: \Delta^k \longrightarrow \Omega^2 X$
$\beta: \Delta^m \longrightarrow \Omega^2 X$

define the cycle
$\lambda_1(\alpha, \beta): \Delta^{k+m+1} \longrightarrow \Omega^2 X$



Can see from picture that $\lambda_k = 0$ on E_{k+2} space