J. Mann talk 2 Fact Alg

Def: Fix $n \in \mathbb{N}$, Mfld_n is cat w. obj n-mfld's M, Mor: $M \xrightarrow{f} N$ of open embedding. Sym mon structure coming from disjoint union

Note: $M \sqcup M \xrightarrow{\nabla} M$ is not in Mfld_n.

Def: $\mathrm{Disk}_n \subset \mathrm{Mfld}_n$ full sub cat on obj's: $\coprod_I \mathbb{R}^n$, $|I| < \infty$

$\mathrm{Disk}_{n/M}$ cat of embedded disks in M.

Rmk: $\mathrm{Disk}_{n/M} \simeq$ Open submfld of M diffeo to unions of $\mathbb{R}^n$'s.

Note: $M \xrightarrow{f} N$ embedding $\rightsquigarrow$ $\mathrm{Disk}_{n/M} \longrightarrow \mathrm{Disk}_{n/N}$.

Def: For $M \in \mathrm{Mfld}_n$ $\mathcal{J}_M \subset \mathrm{Disk}_{n/M}$ w. morph

$$\begin{tikzcd} \coprod_I \mathbb{R}^n \arrow[rr, "f"] \arrow[dr] & & \coprod_J \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

, f isotopic equiv, i.e., $\pi_0 f : I \to J$ is a bijection. And all objects.

Note: $\mathcal{J}_M$ and $\mathrm{Disk}_{n/M}$ have natural filtrations, comes from # of disks.

universal property kinda gadget

very concrete objects

Wonderful Thm [L, AF, ...]: $|\mathcal{J}_M| = \coprod_{k \geq 0} \mathrm{Conf}_k(M)/\Sigma_k$

This should be seen by the faces and degeneracies of $N(\mathcal{J}_M)$ should allow you to wiggle embeddings...

Talk 2

The set of Embeddings have a natural topology, so Disk_n and Mfld_n are topologically enriched.

So denote $\underline{\mathrm{Disk}}_n$ and $\underline{\mathrm{Mfld}}_n$ to be their ass ∞-cats. (Top-enriched, quasi-cat, complete segal, dependent on your needs).

For each $M \in \mathrm{Mfld}_n$: $\mathrm{Disk}_{n/M} \to \underline{\mathrm{Disk}}_{n/M}$ (This trick, in general, allows us to move old school constructions to the fancy world)

Under this functor, maps of J_M goes to equivalences.

<u>Thm</u>: The natural functor $\mathrm{Disk}_{n/M}[J_M^{-1}] \to \underline{\mathrm{Disk}}_{n/M}$ is an equiv

simplicial localization, also, Dwyer, Rezk, ...

In particular it is <u>final</u>.

ex of final functors $\Delta \xrightarrow{\Delta} \Delta \times \Delta$ and $\Delta^{inj} \hookrightarrow \Delta$ which tells us that realization = fat realization.

So again, this is going to allow us to use old categorical tricks in the fancy setting.

Rmk: $(\underline{\mathrm{Disk}}_{n/M})^{\mathrm{core}} \simeq \coprod_k \mathrm{Conf}_k(M)/\Sigma_k$

Maximal subgroupoid.

Recall: Data of a com alg in $\mathcal{C}^{\otimes}$ can be determined as a $\otimes$-functor $\mathrm{FinSets}_* \xrightarrow{A} \mathcal{C}^{\otimes}$

Talk 2

$A(*) = K_{\mathcal{C}^\otimes}$ ← unit of $\mathcal{C}$, $A(* \to \overset{\bullet}{*}) = K_{\mathcal{C}^\otimes} \to A(\overset{\bullet}{*}) =: A$ unit map

$A(\ \ ) = A \otimes A \to A$ multiplication, and $A(\ \ )$ encodes commutativity

Def: A n-disk alg in $\underline{\mathcal{C}}^\otimes$, A, is a $\otimes$-functor $\underline{\mathrm{Disk}}_n \xrightarrow{A} \underline{\mathcal{C}}^\otimes$.

$A(\emptyset) = k_{\mathcal{C}}$, $A := A(\mathbb{R}^n)$, $A(\emptyset \to \mathbb{R}^n)$ gives a unit, and

$\underset{\text{is } \mathcal{O}(n)}{\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)} \to \mathcal{C}(A, A)$ so by adjunction $\mathcal{O}(n) \otimes A \to A$ so

A has a $\mathcal{O}(n)$-action.

So we add geom-decoration to $\mathrm{Emb}^{\square}$ we get closer to little disks. We kill the $O(n)$-action.

The framed E_n-alg is not the same as the framed little disks-alg!

Ex: A a com alg: $\underline{\mathrm{Disk}}_n \xrightarrow{\pi_0(\,)_+} \mathrm{Fin\,Sets}_+ \xrightarrow{A} \mathcal{C}^\otimes$

$\mapsto \quad \mapsto \quad \begin{matrix} A \otimes A \\ \downarrow \\ A \end{matrix}$

Ex: $\mathrm{Conf}_k(\ )_{\mathbb{Z}_k}$ is functorial w.r.t embeddings so an n-disk-alg in Spaces

$$\begin{tikzcd} \mathrm{Disk}_n \arrow[r] \arrow[rr, bend left, "\mathrm{Factors}"] & \underline{\mathrm{Disk}}_n \arrow[r, "\mathrm{Conf}_k(\,)_{\mathbb{Z}_k}"] & \mathrm{Spaces} \arrow[d, "C_*"] \\ & & \mathrm{Ch} \end{tikzcd}$$

Talk 2

"Spinning" two disks inside a larger disk gives

$$S^{n-1} \to \mathrm{Emb}\left(\sqcup_2 \mathbb{R}^n, \mathbb{R}^n\right) \xrightarrow{A} \mathcal{C}(A^{\otimes 2}, A)$$

So assuming $\mathcal{C} = \mathrm{Ch}$ we get $A^{\otimes 2}[n-1] \to A$ which makes A into a shifted Lie-alg (Recall Tyler's talk)

<u>Def</u>: A an n-disk-alg, M - Mfld. $\int_M A = \mathrm{Colim}\left(\underline{\mathrm{Disk}}_{n/M} \to \underline{\mathrm{Disk}}_n \xrightarrow{A} \underline{\mathcal{C}}\right)$

Note: $p_k \in \mathrm{Conf}_k(M)/\Sigma_k \rightsquigarrow * \xrightarrow{p_k} \underline{\mathrm{Disk}}_{n}/M \rightsquigarrow A \to \int_M A$

use the splitting of the core of $\underline{\mathrm{Disk}}_n/M$ to get this map.

given $p_k \xrightarrow{\text{path}} p_k' \rightsquigarrow$
$$\begin{tikzcd} A \arrow[r, bend left, "p_k'"] \arrow[r, bend right, "p_k"'] \arrow[r, phantom, "\Downarrow"] & \int_M A \end{tikzcd}$$