Γ-spaces, Γ-categories, and the Barratt-Priddy Theorem

"Categories and Cohomology Theories" - Graeme Segal

$\boxed{\text{Outline}}$ 1. Γ-spaces

2. Γ-categories

3. Barratt-Priddy Theorem $B\Sigma_\infty \longrightarrow (\Omega^\infty S^\infty)_0$ (conn. comp. of constant loop)

is a homology equivalence

("cohomology theory arising from the category of finite sets (under disjoint union) is stable cohomotopy.")

4. Boardman and Vogt Theorem $O, SO, F, U, PL, Top, \ldots$

their coset spaces F/PL, $PL/O, \ldots$

and their iterated classifying spaces

are infinite loop spaces.

5. Connection with operads

$\boxed{\Gamma\text{-spaces}}$ We will work in compactly generated spaces up to homotopy $\simeq$ - h.e.

Recall $G \rightsquigarrow BG$ $\quad G \underset{\text{w.h.e.}}{\simeq} \Omega BG$

(G top. group, BG top. space)

Γ-space is generalization of a top. ab. group s.t.

$A \rightsquigarrow BA$ (A Γ-space, BA Γ-space)

Def Let Γ be a category with

$ob(\Gamma) =$ finite sets

$mor_\Gamma(S,T) = \{\theta : S \longrightarrow \mathcal{P}(T) \mid \theta(\alpha) \cap \theta(\beta) \neq 0 \Rightarrow \alpha = \beta\}$

composition; $\theta : S \longrightarrow \mathcal{P}(T)$
$\phi : T \longrightarrow \mathcal{P}(U)$ $\quad (\phi \circ \theta)(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \phi(\beta)$

Notation $\underline{n} = \{1, 2, \ldots, n\}$

Def A Γ-space is a functor $A : \Gamma^{op} \longrightarrow Top$ s.t.

(1) $A(\underline{0})$ is contractible

(2) $p_n : A(\underline{n}) \xrightarrow{\simeq} A(\underline{1}) \times \cdots \times A(\underline{1})$

induced by $i_k : \underline{1} \longrightarrow \underline{n}$

Note: $A(\underline{1})$ is "underlying space"

Note: A Γ-space is also a simplicial space

$$\Delta \longrightarrow \Gamma$$

$$[m] \longmapsto \underline{m}$$

$$f:[m]\to[n] \longmapsto \theta(i) = \{ j \in \underline{n} \mid f(i-1) < j \leq f(i) \}$$

Note: If p_n's are all iso's, then A is a topological abelian monoid (group w/o inverses)

$$\Gamma\text{-space} \longrightarrow \text{spectra}$$

Def $(BA)(S) = |A(S \times -)|$

$\mathbb{B}A$ is a spectrum $\overset{S=\underline{1}}{}$ $A(\underline{1}), BA(\underline{1}), B^2A(\underline{1}), \ldots$

$$\mathbb{B}A_n = B^n A(\underline{1})$$

$$\Sigma A(\underline{1}) \longrightarrow |A| = BA(\underline{1})$$

hom. eq. onto 1-skeleton

Modified Realization $A = \{A_n\}$ simplicial space

$$|A| = \left(\coprod_{n \geq 0} \Delta^n \times A_n \right) \Big/ \left((\underset{\Delta^m}{\xi}, \underset{A_n}{\theta^* a}) \sim (\theta_* \xi, a) \right) \quad \theta : [m] \to [n] \in \Delta$$

has problems:
1. ~~Produces spaces~~ Takes you out of W (spaces w/ hom. type of CW-cxs.)
2. $A \to A'$ w/ $A_n \xrightarrow{\simeq} A_n'$ $\forall n$ $\not\Longrightarrow$ $|A| \xrightarrow{\simeq} |A'|$

$\|A\|$ — only kill face maps, not degeneracy maps $\left(\|A\| = \left| [n] \mapsto \coprod_{\sigma \subset n} A_{n,\sigma} \right| \right)$

Properties of $\|A\|$

1. $A_n \in W$ $\forall n \Longrightarrow \|A\| \in W$
2. $A \to A'$ w/ $A_n \xrightarrow{\simeq} A_n'$ $\forall n \Longrightarrow \|A\| \xrightarrow{\simeq}$
3. $\|A \times A'\| \simeq \|A\| \times \|A'\|$
4. For A good ($A_{n,i} \hookrightarrow A_n$ closed cofibration), $\|A\| = |A|$

$\boxed{\Gamma\text{-categories}}$ Assume all categories are topological categories
(set of objects & set of morphisms have topologies s.t. structural maps are continuous)

Cat $\rightsquigarrow$ Γ-cat $\longrightarrow$ Γ-space $\rightsquigarrow$ spectrum

Def A Γ-category is a functor $\mathcal{C}: \Gamma^{op} \to Cat$ s.t.

1. $\mathcal{C}(\underline{0})$ is equiv. to cat. w/ 1 obj. & 1 morph.
2. $p_n : \mathcal{C}(\underline{n}) \xrightarrow{\simeq} \mathcal{C}(\underline{1}) \times \dots \times \mathcal{C}(\underline{1})$ induced by $i_k : \underline{1} \to \underline{n}$

Note
Γ-cat w/ 2 comp. laws or Γ-space w/ a multiplications $\rightsquigarrow$ ring spectrum

Cor $\mathcal{C}$ is Γ-cat. $\Longrightarrow$ $\|\mathcal{C}\|$ is Γ-space

Ex 1 $(\mathcal{C},+)$ is a cat. w/ sums $\Longrightarrow$ $\mathcal{C}(S)$ is cat.

finite set

$ob_{\mathcal{C}(S)} = \{Fun(\mathcal{P}(S), \mathcal{C}) \mid \sqcup \text{ goes to } +\}$

$mor_{\mathcal{C}(S)}$ = isomorphisms of functors

$\mathcal{C}(\underline{0}) = \mathrm{id}_+$

$\mathcal{C}(\underline{1}) = \mathcal{C}$

$\mathcal{C}(\underline{2}) \ni A_1 \longrightarrow A_{12} \longleftarrow A_2$

in $\mathcal{C}$ w/ univ. prop. expressing $A_{12} = A_1 + A_2$

Why Γ-cat.?

$\mathcal{C}(\underline{2}) \xrightarrow{\simeq} \mathcal{C} \times \mathcal{C}$

$A_1 \to A_{12} \leftarrow A_2 \longmapsto (A_1, A_2)$ ~~is equiv. of cat~~

$mor_\Gamma(S,T) = F(\mathcal{P}(S), \mathcal{P}(T))$ that preserve disjoint union

$\theta : S \to \mathcal{P}(T)$

$\theta(\alpha) \cap \theta(\beta) \neq 0 \Rightarrow \alpha = \beta$

Ex 2 replace "sum" w/ "product" to get $\mathcal{C}^{\Pi}(S)$

Ex 3 $\mathcal{C}$ = modules over a commutative ring $\rightsquigarrow \mathcal{C}^{\otimes}(S)$

$\mathcal{C}^{\otimes}(\underline{2}) \ni (M_1, M_2, M_{12}, \alpha_{12})$ $\alpha_{12} : M_1 \times M_2 \longrightarrow M_{12}$ bilinear sat. univ. prop. for tensor product

Barratt-Priddy Theorem

$\mathcal{A}$ = cat. of Γ-spaces and hom.-classes of weak morphisms $A \to A'$ $(A \xleftarrow{\simeq} \bar{A} \to A'$

$\mathcal{S}\mu$ = cat. of spectra and hom.-classes of morphisms

$$\mathbb{B} : \mathcal{A} \underset{\text{adjoint}}{\rightleftarrows} \mathcal{S}\mu : \mathbb{A}$$

<u>Def</u> If X is a spectrum, then $\mathbb{A}X$ is a Γ-space defined by

$$\mathbb{A}X(\underline{n}) = \operatorname{Mor}(\underbrace{\$ \times \dots \times \$}_{n} ; X)$$

<u>Barratt-Priddy Theorem</u> $\mathbb{B}(B\Sigma) \simeq \$$

<u>Proof</u> Suffices to show

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{S}\mu}(\mathbb{B}(B\Sigma); X) & \simeq & \operatorname{Hom}_{\mathcal{S}\mu}(\$; X) \\ \| & & \| \\ \operatorname{Hom}_{\mathcal{A}}(B\Sigma; \mathbb{A}X) & & \pi_0(X) \\ & & \| \\ & & \pi_0(\mathbb{A}X(\underline{1})) \end{array}$$

So, suffices to show $\operatorname{Hom}_{\mathcal{A}}(B\Sigma; A) = \pi_0(A(\underline{1}))$

$$B\Sigma \to A \longmapsto \text{conn. comp. of image of } B\Sigma_1 \in B\Sigma(\underline{1}) = \coprod_{n \geq 0} B\Sigma_n$$

$\mathcal{S}$ = category of finite sets under disjoint union

<u>Def</u> Define the Γ-space $B\Sigma$ by $B\Sigma(\underline{n}) = \| \mathcal{S}(\underline{n}) \|$

$$B\Sigma(\underline{1}) = \coprod_{n \geq 0} B\Sigma_n$$

$$B\Sigma(\underline{2}) = \coprod_{m,n \geq 0} (E\Sigma_m \times E\Sigma_n \times E\Sigma_{m+n}) / (\Sigma_m \times \Sigma_n)$$

$$B\Sigma(\underline{k}) = \coprod_{m_1, \dots, m_k \geq 0} \left(\prod_{\sigma \subset \underline{k}} E\Sigma_{m_\sigma} \right) / (\Sigma_{m_1} \times \dots \times \Sigma_{m_k})$$

$$m_\sigma = \sum_{\alpha \in \sigma} m_\alpha$$

Boardman and Vogt Theorems

Prop $\mathbb{B}A_k = B^k(A)(\underline{1}) \simeq \Omega B^{k+1}(A)(\underline{1}) = \mathbb{B}A_{k+1} \quad \forall k \geq 1$
and for $k=0$ iff the H-space $A(1)$ has a homotopy inverse

Whenever we can naturally associate $G(S)$ (top. monoid, instead of Σ_n, containing $\Sigma(S)$) to each finite set S
and associative nat. transformations $G(S) \times G(T) \longrightarrow G(S \amalg T)$

Relation to Operads

spaces w/ an operad action $\rightsquigarrow$ Γ-spaces

$K \curvearrowright X_{\text{space}} \approx A : K^{op} \to Top$

space of objects is discrete — {category of operators w/ ob_K = nat. numbers

$\pi : K \xrightarrow{\simeq} \Gamma$

$\pi_*(A)$

$\pi_*(A)(s)$
$\parallel$
realization of simplicial space whose k-simplices are

$$\coprod_{T_0, \ldots T_k \in ob(K)} Mor_{\Gamma}(S; \pi T_0) \times Mor_K(T_0; T_1) \times \ldots \times Mor_K(T_{k-1}; T_k) \times A(T_k)$$

Γ-spaces $\rightsquigarrow$ space w/ an operad action

???