Operads and Loop Space Machinery Seminar

Week 1: Exercises

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(1) A pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ form an *adjunction* if there is a natural family of bijections

$$hom_{\mathcal{C}}(GX, Y) \cong hom_{\mathcal{D}}(X, FY),$$

for all $X \in \mathcal{D}$ and $Y \in \mathcal{C}$.

(a) Show that $\Sigma : Top_* \leftrightarrows Top_* : \Omega$ is an adjunction.

(b) Show that $\Sigma^n : Top_* \leftrightarrows Top_* : \Omega^n$ is an adjunction.

(c) Show that $\Sigma^\infty : Top_* \leftrightarrows \Omega - Spectra : \Omega^\infty$ is an adjunction.

(2) We will need the notion of simplicial objects and geometric realization later. A *simplicial object* in a category $\mathcal{C}$ is a sequence of objects $X_q \in \mathcal{C}$, together with *face maps*

$$\partial_i : X_q \to X_{q-1}$$

and *degeneracy maps*

$$s_i : X_q \to X_{q+1}$$

for $0 \leq i \leq q$, such that the *simplicial identities* hold:

- $$\partial_i \partial_j = \partial_{j-1} \partial_i \quad if \quad i < j,$$
- $$\partial_i s_j = \begin{cases} s_{j-1}\partial_i & \text{if } i < j, \\ 1 & \text{if } i = j \text{ or } i = j+1, \\ s_j \partial_{i-1} & \text{if } i > j+1, \end{cases}$$
- $$s_i s_j = s_{j+1} s_i \quad if \quad i \leq j.$$

A morphism of simplicial objects $\{X_q\} \to \{Y_q\}$ is just a sequence of morphisms $f_q : X_q \to Y_q$ which commute with the structure maps. The category of simplicial objects in $\mathcal{C}$ is denoted $s\mathcal{C}$.

If $X_\bullet \in sSets_*$, then we can define the *geometric realization* of $X_\bullet$ by setting

$$|X| := \left(\coprod_{q=0}^{\infty} X_q \times \Delta^q \right) / \sim$$

where $\sim$ is defined by $(x, p) \sim (y, q)$ if either

- $\partial_i x = y$ and $d^i q = p$, or
- $s_j x = y$ and $s^j q = p$

where d^i, s^j are the usual face and degeneracy maps for the standard q-simplex $\Delta^q \subset \mathbb{R}^{n+1}$.

(a) (Lemma 9.2 of May) Let $X \in \mathcal{C}$.

(i) Show that one can define a simplicial object $X_\bullet \in s\mathcal{C}$ by setting $X_q = X$ and $\partial_i, s_j = id$ for all i, j.

(ii) If $Y_\bullet \in s\mathcal{C}$, determine show that a map $f : X \to Y_0$ determines and is determined bythe map $\tau_\bullet(f) : X_\bullet \to Y_\bullet$ defined by $\tau_q(f) = s_0^q f$.

(iii) Show that a map $g : Y_0 \to X$ such that $g\partial_0 = g\partial_1$ determines and is determined by the map $\epsilon_\bullet(g) : Y_\bullet \to X_\bullet$ with $\epsilon_q(g) = g\partial_0^q$.

(b) Show that there is an equivalence of categories

$$|-| : sSets_* \leftrightarrows Top_* : Sing_\bullet(-).$$

(3) A simplicial space is just a simplicial object in unbased topological spaces. Let $X_\bullet$ be a simplicial space. Let $sX_q = \bigcup_{j=0}^q s_j X_q \subset X_{q+1}$. We say X is *proper* if (X_{q+1}, sX_q) is a strong NDR-pair for all q. We say X is *strictly proper* if each $(X_{q+1}, s_k X_q)$ is a strong NDR pair via a homotopy $h : I \times X_{q+1} \to X_{q+1}$ such that

$$h(I \times \bigcup_{j=0}^{k-1} s_j X_q) \subset \bigcup_{j=0}^{k-1} s_j X_q.$$

Given a simplicial space, the formulas above naturally define a geometric realization functor

$$|-| : sTop \to Top.$$

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a simplicial map between strictly proper simplicial spaces. Assume that each f_q is a weak homotopy equivalence, and either $|X|$ or $|Y|$ are simply connected or that $|f|$ is an H-map between connected H-spaces. Then $|f|$ is a weak homotopy equivalence.