Operads and Loop Space Machinery Seminar
Week 2: Minicourse Part II, Operads
J.D. Quigley

1. References.

This part of the minicourse will be based on Sections 1-3 of May's "The geometry of iterated loop spaces". The exposition will also draw from Adams' "Infinite loop spaces" and May's survey "Infinite loop space theory".

2. Motivation and definition of operads

Our goal is to understand when a space X is actually an n-fold or infinite loop space. First, let's try to understand some of the additional structure we have in an n-fold loop space.

Example 2.1. *Say X is a 1-fold loop space, so $X \cong \Omega Y$ for some Y. Then there is a natural product structure on X, i.e. a map $X \times X \to X$, induced by the composition of loops:*

$$X \times X \cong \Omega Y \times \Omega Y \to \Omega Y \cong X$$
$$(g, f) \mapsto g \circ f.$$

Note that we made a choice here! Composition of loops is usually defined by running the first loop and second loop at equal speeds, i.e. dividing the unit interval at $\frac{1}{2}$. However, we could have defined a product by choosing any real number $0 < i_1 < 1$. Therefore we actually have an entire interval worth of products, all of which are equivalent up to homotopy (by reparametrizing the loops).

The complexity of this space of products increases when we assume X is a 2-fold loop space.

Example 2.2. *Say X is a 2-fold loop space, so $X \cong \Omega(\Omega Y) \cong \Omega^2 Y$ for some Y. As above, there is a product on X*

$$X \times X \cong \Omega\Omega Y \times \Omega\Omega Y \to \Omega\Omega Y \cong X$$
$$(g_2 \circ g_1, f_2 \circ f_1) \mapsto g_2 \circ g_2 \circ f_2 \circ f_1$$

Then there is a larger space of products corresponding to the different ways of parametrizing these compositions, all of which are equivalent up to homotopy. Moreover, these products should be compatible with the products we obtain if we forget that X is a 2-fold loop space and only think of it as a 1-fold loop space.

In general, the more "deloopable" X is, the more complicated the space of (equivalent up to homotopy) products on X becomes. An operad is a tool which keeps track of these spaces of products and their compatibilities with each other.

Definition 2.3. An *operad* $\mathcal{O}$ consists of spaces $\mathcal{O}(j) \in Top$ for $j \geq 0$ with $\mathcal{O}(0) = pt$, together with continuous functions

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \to \mathcal{O}(j)$$

where $j = \sum j_s$, such that the following properties hold:

(1) (associativity) For all $c \in \mathcal{O}(k)$, $d_s \in \mathcal{O}(j_s)$, and $e_t \in \mathcal{O}(i_t)$,
$$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_j) = \gamma(c, f_1, \ldots, f_k),$$
where $f_s = \gamma(d_s; e_{j_1+\cdots+j_{s-1}+1}, \ldots, e_{j_1+\cdots+j_s})$ and $f_s = pt$ if $j_s = 0$.

(2) (identity) There is an element $1 \in \mathcal{O}(1)$ such that $\gamma(1; d) = d$ for all $d \in \mathcal{O}(j)$ and $\gamma(c; 1, \ldots, 1) = c$ for $c \in \mathcal{O}(k)$

(3) (equivariance) A right action of the symmetric group Σ_j on $\mathcal{O}(j)$ such that for all $c \in \mathcal{O}(k)$, $d_s \in \mathcal{O}(j_s)$, $\sigma \in \Sigma_k$, and $\tau_s \in \Sigma_{j_s}$,
$$\gamma(c\sigma; d_1, \ldots, d_k) = \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)})\sigma(j_1, \ldots, j_k),$$
$$\gamma(c; d_1\tau_1, \ldots, d_k\tau_k) = \gamma(c; d_1, \ldots, d_k)(\tau_1 \oplus \cdots \oplus \tau_k),$$
where $\sigma(j_1, \ldots, j_k)$ denotes the permutation of j letters which permutes the k blocks of letters determined by the given partition of j as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k$ denotes the image of $(\tau_1, \ldots, \tau_k)$ under the inclusion of $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k}$ in Σ_j.

An operad $\mathcal{O}$ is *Σ-free* if Σ_j acts freely on $\mathcal{O}(j)$ for all j. A *morphism of operads* is a sequence of Σ_j-equivariant maps which commute with γ.

In general, we think of the j-th space of an operad as encoding the j-ary operations. We'll work through the following examples during the exercises this week. For now, we'll just the spaces and maps without checking the axioms. The first two examples can be thought of as encoding structural information; in particular, they encode strictly commutative multiplication and strictly associative multiplication.

Example 2.4. (1) *The* commutative operad *$\mathcal{C}om$ has spaces $\mathcal{C}om(j) = *$. There is only one choice for the structure maps γ. In this case, the operad says that multiplying in any order and even switching factors does not change the output.*

(2) *The* associative operad *$\mathcal{A}s$ has spaces $\mathcal{A}s(j) = \Sigma_j$. In this case, we are allowed to change the order of multiplication, but not switch factors.*

3. Endomorphism operads and $\mathcal{O}$-spaces

We would like to have a way of talking about an operad encoding multiplication on a space. The correct notion here is an operad acting on a space; drawing out all of the diagrams for this to make sense is somewhat tedious, so instead this idea is repackaged in the following example/definition.

Definition 3.1. Let X be a based space. The *endomorphism operad $\mathcal{E}_X$ of X* is defined as follows. Let $\mathcal{E}_X(j)$ be the space of based maps $X^j \to X$, with $X^0 = *$, and $\mathcal{E}_X(0)$ the inclusion $* \to X$. The structure maps are

(1) $\gamma(f; g_1, \ldots, g_k) = f(g_1 \times \cdots \times g_k)$ for $f \in \mathcal{E}_X(k)$ and $g_s \in \mathcal{E}_X(j_s)$.
(2) Identity element $1 \in \mathcal{E}_X(1)$ is the identity map $X \to X$
(3) $(f\sigma)(y) = f(y\sigma)$ for $f \in \mathcal{E}_X(j)$, $\sigma \in \Sigma_j$, and $y \in X^j$.

An *operation* θ of an operad $\mathcal{O}$ on a space X is a morphism of operads $\theta : \mathcal{O} \to \mathcal{E}_X$. The pair (X, θ) is called an $\mathcal{O}$-space. A *morphism of* $\mathcal{O}$*-spaces* $f : (X, \theta) \to (X', \theta')$ is just a map of spaces $X \to X$ which commutes with the operations.

We'll work through an explicit example in the exercise session this week, but we'll include some details for the most topologically simple operad now.

Example 3.2. *To get a better idea of what an* $\mathcal{O}$*-space is, we make explicit some of the structure of an* $\mathcal{O}$*-space* (X, θ) *for the commutative operad* $\mathcal{C}om$*. Firstly, a morphism of operads is a sequence of equivariant maps which commute with the structure maps* γ*. In this case, we have* Σ_j*-equivariant maps*

$$\theta(j) : \mathcal{C}om(j) = * \to Map_*(X^j, X) = \mathcal{E}_X(j).$$

Since the identity in $\mathcal{C}om(1)$ *must map to the identity in* $\mathcal{E}_X(1)$*, we have that*

$$\theta(1)(*) = (id : X \to X) \in Maps_*(X, X) = \mathcal{E}_X(1).$$

Let's examine $\theta(j)$ *for* j *small. Let* γ_C *be the structure map for the commutative operad and let* γ_X *be the structure map for the endomorphism operad of* X*. For* $k = 2$*,* $j_1 = j_2 = 1$*, we have a commutative diagram*

$$\begin{tikzcd}
* \times * \times * \arrow[r, "\gamma_C"] \arrow[d, "\mu \times id \times id"] & * \arrow[d, "id^2"] \\
Maps_*(X^2, X) \times Maps_*(X, X) \times Maps_*(X, X) \arrow[r, "\gamma_X"] & Maps_*(X^2, X).
\end{tikzcd}$$

The Σ_2*-action on the top arrow says that we can permute the two right-most points without changing the effect. Mapping this permutation downwards in the diagram, we conclude that in the bottom row,*

$$\gamma_X(\theta(2)(*); id, id) = \gamma_X(\theta(2)(*); id, id).$$

Denoting $\theta(2)(*)$ *by* $\mu : X \times X \to X$*, we see that* $\theta(2)$ *is picking out a strictly comutative multiplication on* X*, i.e.* $\mu(x, y) = \mu(y, x)$ *for all* $x, y \in X$*.*

Similar arguments show that $\theta(3)$ *picks out ways of multiplying three elements, and by compatibility with* $\theta(2)$*, we obtain the relation*

$$\mu(x, \mu(y, z)) = \mu(\mu(x, y), z).$$

Since μ *is commutative, we conclude that* $\theta(3)$ *must pick out a way of multiplying three elements in* X *such that the order doesn't matter, and so that the choice is consistent with the product on two elements.*

Going further, we would obtain ways of multiplying any number of elements, with the requirement that we can multiply in any order and the n*-fold multiplication is compatible with the* i*-fold multiplications for all* $2 \leq i \leq n - 1$*.*

4. The Recognition Principle and A_∞ and E_∞ operads

Some thought shows that $\mathcal{C}om$-spaces are too rigid to model infinite loop spaces. Indeed, in order to define a multiplication on loop spaces, we had to choose a parametrization for composing loops, and this choice will not give equivalent loops when we try to compose more than two loops. For example, if we decided that $\mu(f,g)$ is just $f \circ g$ where we run both loops at the same speed, then $\mu(f,\mu(g,h)) \neq \mu(\mu(f,g),h)$. Therefore an infinite loop space cannot naturally be given the structure of a $\mathcal{C}om$-space. The correct operad to encode the structure of an infinite loop space is given by the Recognition Principle.

Theorem 4.1 (Recognition Principle)**.** *There exists Σ-free operads $\mathcal{O}_n$, $1 \leq n \leq \infty$, such that every n-fold loop space is a $\mathcal{O}_n$-space and every connected $\mathcal{O}_n$-space has the weak homotopy type of an n-fold loop space.*

At the extremes, we have two important definitions. A 1-fold loop space has a product which we should expect to be associative up to homotopy (and all higher homotopies). On the other hand, an infinite loop space has a product which we should expect to be commutative up to homotopy (and all higher homotopies). In this case, the operads $\mathcal{O}_n$ can be replaced by A_∞ and E_∞ operads.

Definition 4.2. An A_∞ *operad* is a Σ-free operad $\mathcal{O}$ such that $\mathcal{O}(j)$ is Σ_j-equivariantly homotopy equivalent to the space Σ_j.

An E_∞ *operad* is a Σ-free operad $\mathcal{O}$ such that $\mathcal{O}(j)$ is Σ_j-contractible for all j.

The definition above is formulated so that there isn't just *one* A_∞ or E_∞ operad. In fact, there are many A_∞ and E_∞ operads. In the context of infinite loop spaces, we are interested in E_∞ operads. We will see that certain E_∞ operads naturally act on some infinite loop spaces but not others. This flexibility in choice of E_∞ operad is the real strength of this framework and the Recognition Principle.

Example 4.3. *We begin by listing various A_∞ operads.*

1. *The associative operad $\mathcal{A}s$ is an A_∞ operad.*
2. *A combinatorial model of an A_∞ operad is given by* Stasheff associahedra*. We will not discuss it further, but we mention that this operad plays an important role in Robinson's A_∞ obstruction theory.*
3. *The model closest to what we will see in the exercises and the proof of the Recognition Principle is the* little intervals operad $\mathcal{I}$*, defined by setting the n-th level*

$$\mathcal{I}(n) = Emb\left(\coprod_{i=1}^{n} I, I\right)$$

to be embeddings of n disjoint intervals into the unit interval. This operad naturally appears in the study of configuration spaces, as well as functor calculus.

Example 4.4. *We now list some examples and non-examples of E_∞ operads.*

(1) *The commutative operad* $\mathcal{C}om$ *is not* E_∞ *operad, since the action of* Σ_n *on* $\mathcal{C}om(n) = *$ *is not free.*

(2) *The* Barratt-Eccles operad $\mathcal{E}$ *is defined by setting the* n*-th level*

$$\mathcal{E}(n) = E\Sigma_n$$

to be the universal space for Σ_n. *We will not discuss it further, but we mention that this operad plays an important role in Robinson's* E_∞ *obstruction theory.*

(3) *The* little n-cubes operads $\mathcal{O}_n$ *are the operads which appear in the statement of the Recognition Principal. The levels of* $\mathcal{O}_n$ *are defined as follows. Let* $J^n = int(I^n)$ *be an open* n*-cube. An* (open) little n-cube *is a linear embedding of* J^n *in* J^n. *Let* $\mathcal{O}_n(j)$ *to be the space of pairwise disjoint ways of embedding* j *little* n*-cubes into* J^n. *Here, the topology is the subspace topology coming from the compact-open topology on* $Emb((J^n)^j, J^n)$. *Letting* n *tend to infinity gives an* E_∞*-operad.*

We will explore these definitions and examples further in the exercise session this week.