Operads and Loop Space Machinery Seminar

Week 3: Minicourse Part III, The Approximation Theorem

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1. References.

This part of the minicourse will be based on Sections 2-6 of May's "The geometry of iterated loop spaces". We are also grateful to Jonathan Rubin and Jens Jakob Kjaer for clarifying some of this material.

2. Monoidal categories and monads

The Recognition Principle follows by reduction to the Approximation Theorem. In order to state the Approximation Theorem, we need the notion of *monads*. We'll give a streamlined definition using category theory this week, but during the exercises we'll unpack some of these definitions.

Definition 2.1. A *monoidal category* $(\mathcal{C}, \otimes, 1)$ consists of the following data:

- A category $\mathcal{C}$.
- An associative and unital multiplication $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying some natural compatibility axioms
- An object $1 \in \mathcal{C}$ which is a unit for the multiplication

Roughly speaking, a monoidal category $(\mathcal{C}, \mu, 1)$ is *symmetric monoidal* if the multiplication is commutative.

A *monoid* in a monoidal category $(\mathcal{C}, \mu, \eta)$ is an object $M \in \mathcal{C}$ together with morphisms $\mu : M \otimes M \to M$ and $\eta : 1 \to M$, such that certain compatibility diagrams commute.

Example 2.2. *We already know several examples of (symmetric) monoidal categories:*

1. *The category of nonempty pointed sets has unit object some singleton set and multiplication given by Cartesian product. A monoid in this setting is a group.*
2. *The category of pointed topological spaces has unit object a point and multiplication given by Cartesian product. A loop space is an example of a monoid in this setting.*
3. *The category of rings has unit object the integers and multiplication given by tensor product. A monoid in this setting is an algebra.*

To define monads, we'll introduce another monoidal category.

Definition 2.3. Let $\mathcal{C}$ be a category. The *category of endofunctors of* $\mathcal{C}$, denoted $End_{\mathcal{C}}$, has the following data:

1. Objects are endofunctors of $\mathcal{C}$, i.e. functors $F : \mathcal{C} \to \mathcal{C}$.
2. Morphisms are natural transformations of functors, i.e. $\eta : F \Rightarrow G$.
3. Composition is given by composition of natural transformations.

This is a monoidal category with unit object the identity functor $id : \mathcal{C} \to \mathcal{C}$ and multiplication given by composition of functors,

$$\mu : End_{\mathcal{C}} \times End_{\mathcal{C}} \to End_{\mathcal{C}},$$
$$(F, G) \mapsto F \circ G.$$

A *monad* is a monoid in $End_{\mathcal{C}}$.

This concept packages a lot of information we want into one definition. Furthermore, monads naturally occur any time we have an adjunction.

Example 2.4. *Suppose we have an adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G.$$

Then the composition $G \circ F : \mathcal{C} \to \mathcal{C}$ is a monad. We include the verification as an optional exercise this week.

We can apply this fact to the adjunctions

$$\Sigma^n : Top_* \rightleftarrows Top_* : \Omega^n$$

to obtain a monad $\Omega^n \Sigma^n$.

In the limiting case, we have an adjunction

$$\Sigma^\infty : Top_* \rightleftarrows \Omega - Spectra : \Omega^\infty$$

which gives a monad $Q := \Omega^\infty \Sigma^\infty$.

3. Monads from operads and the Approximation Theorem

Now, we return to operads. Just as with an adjunction, an operad always has an associated monad. To define this monad, we need to produce certain maps σ_i between space in an operad; one example of these maps is given below.

Example 3.1. *In the endomorphism operad of a space X, there is a natural map $\sigma_i : \mathcal{E}_X(j) \to \mathcal{E}_X(j-1)$ induced by evaluating on the basepoint of the i-th factor of X^j. That is, we define*

$$s_i : X^{j-1} \to X^j,$$
$$\overline{x} = (x_1, \dots, x_{j-1}) \mapsto (x_1, \dots, x_i, *, x_{i+1}, \dots, x_{j-1})$$

and for $c \in \mathcal{E}_X(j) = Maps_*(X^j, X)$ and overlinex $\in X^{j-1}$, we set

$$(\sigma_i c)(\overline{x}) = c(s_i(\overline{x})).$$

The structure maps in an operad allows us to define the map $\sigma_i : \mathcal{O}(j) \to \mathcal{O}(j-1)$ more generally. For $c \in \mathcal{O}(j)$, let $\sigma_i c = \gamma(c; s_i)$ where

$$s_i = 1^i \times * \times 1^{j-i-1} \in \mathcal{O}(1)^i \times \mathcal{O}(0) \times \mathcal{O}(1)^{j-i-1}.$$

The purpose of these maps is to rectify the fact that the unit in our operads is often only a unit up to homotopy; essentially, the maps σ_i glue together the different choices of unit when multiplying. We now define the monad C associated to an operad $\mathcal{O}$; if we evaluate C on a space X, the space CX is the “free $\mathcal{O}$-algebra on X”.

Construction 3.2. *Let $\mathcal{O}$ be an operad. We can construct a monad (C, μ, η) associated to $\mathcal{O}$ as follows. For a pointed space X, let*

$$CX = \left(\coprod_{j \geq 0} \mathcal{O}(j) \times X^j \right) / \sim$$

where

$$(\sigma_i c, y) \sim (c, s_i y) \quad for \quad c \in \mathcal{O}(j), 0 \leq i < j, y \in X^{j-1},$$
$$(c\sigma, y) \sim (c, \sigma y) \quad for \quad c \in co(j), \sigma \in \Sigma_j, y \in X^j.$$

We will not differentiate between an element in $\mathcal{O}(j) \times X^j$ and its image in CX. The monad structure maps are given by

$$\mu : C^2X \to CX,$$
$$\mu(c, (d_1, y_1), \ldots, (d_k, y_k)) = (\gamma(c; d_1, \ldots, d_k), y_1, \ldots, y_k),$$

and

$$\eta : X \to CX,$$
$$\eta(x) = (1, x).$$

There is an obvious filtration on CX coming from truncating the disjoint union at some finite index, i.e.

$$F_n CX = \left(\coprod_{j=0}^{n} \mathcal{O}(j) \times X^j \right) / \sim .$$

In fact, the topology on CX is defined as the colimit topology of the topologies on these pieces, which have the quotient topology. The associated graded pieces of this filtration are homeomorphic to the levels of the *monadic bar construction* of $\mathcal{O}$ and X which we will discuss next week. If X is an infinite loop space, then the Recognition Principle tells us that X receives an action of an E_∞ operad. If we choose the operad to be the Barratt-Eccles operad, then the homology of $F_n/F_{n-1}CX$ is easy to understand in terms of the homology of X, and structure map $CX \to X$ can be used to define homology operations.

We can now state the Approximation Theorem:

Theorem 3.3. *For the operads $\mathcal{O}_n$ of the Recognition Principle, $1 \leq n \leq \infty$, there is a natural map of $\mathcal{O}_n$-spaces*

$$\alpha_n : C_n X \to \Omega^n \Sigma^n X$$

which is a weak homotopy equivalence if X is connected.

Using Exercise 4 from last week and the structure maps for a monad, we can already define the map α_n. It is the composite

$$C_n X \xrightarrow{C_n \eta_n} C_n \Omega^n \Sigma^n X \xrightarrow{\theta_n} \Omega^n \Sigma^n X,$$

where η_n is the unit of the monad $\Omega^n \Sigma^n$. In fact, we can say more about this morphism of monads.

It is easy to check that if $F : \mathcal{C} \leftrightarrows \mathcal{C} : G$ form an adjunction and C is a monad in $\mathcal{C}$, then FCG is a monad in $\mathcal{C}$ as well. With this in mind, we have the following factorization of α.

Proposition 3.4. *For $n > 1$, there is a morphism of monads $\beta_n : C_n \to \Omega C_{n-1}\Sigma$ such that $\alpha_n = (\Omega\alpha_{n-1}\Sigma)\beta_n$. Therefore α_n factors as a composite of morphism of monads*

$$C_n \to \Omega C_{n-1}\Sigma \to \cdots \to \Omega^{n-1}C_1\Sigma^{n-1} \to \Omega^n\Sigma^n.$$

The proof is similar to the proof of Exercise 4 from Week 2 in the sense that one must define β_n explicitly by using the geometry of the little n-cubes and little $(n-1)$-cubes operads. We will leave the definition of this map and the proof of the proposition for the exercises this week.

4. Proof of the Approximation Theorem

The Approximation Theorem follows from the following commutative diagram:

$$\begin{array}{ccccc}
C_nX & \xrightarrow{\subset} & E_nX & \xrightarrow{\pi_n} & C_{n-1}\Sigma X \\
\downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \tilde{\alpha}_n} & & \downarrow{\scriptstyle \alpha_{n-1}} \\
\Omega^n\Sigma^n X & \xrightarrow{\subset} & P\Omega^{n-1}\Sigma^n X & \xrightarrow{P} & \Omega^{n-1}\Sigma^n X.
\end{array}$$

Note that α_0 is just the identity map. The properties of the diagram we need are the following:

(1) The bottom row is just the path-loops fibration.
(2) There exist contractible spaces E_nX, for all $n \geq 1$, which make the diagram commute.
(3) The map π_n is a quasi-fibration with fiber C_nX when X is connected.

The first and third properties say that we get long exact sequences in homotopy. The contractibility in the second property says that the homotopy groups of E_nX vanish. The five lemma and induction on n then complete the proof. We will construct the diagram in the remainder of the lecture. The proofs of the second two properties are the content of Section 7 of May's "The geometry of iterated loop spaces".

Construction 4.1. *We now define the functor $E_n(-,-)$ from pairs of based spaces to spaces; the space $E_n(X)$ is defined to be $E_n(cone(X), X)$. Let $\mathcal{E}_n(j; X, A)$ be the subspace of $\mathcal{O}_n(j) \times X^j$ consisting of all points $(\langle c_1, \ldots, c_j\rangle, x_1, \ldots, x_j)$ such that if $x_r \notin A$, then the intersection in J^n of the sets $(c_r'(0), 1) \times c_r"(J^{n-1})$ and $c_s(J^n)$ is empty for all $s \neq r$. Here, the prime notation indicates we are thinking of c_r as a product $c_r' \times c_r" : J^n = J \times J^{n-1} \to J^n$. We then set*

$$E_n(X, A) := \bigoplus_{j\geq 0} \mathcal{E}_n(j; X, A)/\sim,$$

where $\sim$ is the restriction of the equivalence relation in Construction 3.2. The topology on $E_n(X, A)$ is the subspace topology coming from the inclusion $E_n(X, A) \hookrightarrow C_nX$. Note that there is also a map $C_nA \hookrightarrow E_n(X, A)$.

If we apply $E_n(-)$ to the pair $(X, *)$, we can compare to $C_{n-1}X$ via the following map.

Proposition 4.2. *There is a natural surjective based map $v_n : E_n(X, *) \to C_{n-1}X$ defined on $[c, x] \in E_n(X, *)$ by*

$$v_1(c, x) = x \in X = C_0X,$$
$$v_n(c, x) = (c'', x) \in C_{n-1}X \quad if \quad n > 1.$$

If $\pi : (X, A) \to (Y, *)$ is a map of pairs, then we can define π_n to be the composite

$$E_n(X, A) \xrightarrow{E_n\pi} E_n(Y, *) \xrightarrow{v_n} C_{n-1}Y.$$

Applying this in the case $\pi : (cone(X), X) \to (\Sigma X, *)$, we obtain the top row of the main commutative diagram:

$$C_nX \xrightarrow{\subset} E_n(TX, X) \xrightarrow{\pi_n} C_{n-1}\Sigma X.$$

There is a map

$$\tilde{\eta_n} : TX \to P\Omega^{n-1}\Sigma^nX,$$

defined by setting

$$[\tilde{\eta_n}(x, s)](t)(v) = [x, st, v]$$

for $[x, s] \in TX$, $t \in I$, and $v \in S^{n-1}$. Applying $E_n(-)$ to both sides of $\tilde{\eta_n}$, we obtain a commutative diagram

$$\begin{array}{ccccc} C_nX & \xrightarrow{\subset} & E_nX & \xrightarrow{\pi_n} & C_{n-1}\Sigma X \\ \downarrow{\scriptstyle C_n\eta_n} & & \downarrow{\scriptstyle E_n(\tilde{\eta_n})} & & \downarrow{\scriptstyle C_{n-1}\eta_{n-1}} \\ C_n\Omega^n\Sigma^nX & \xrightarrow{\subset} & E_n(P\Omega^{n-1}\Sigma^nX) & \xrightarrow{P} & C_{n-1}\Omega^{n-1}\Sigma^nX. \end{array}$$

We can obtain the main commutative diagram from this by composing the left and right vertical arrows with θ_n and θ_{n-1}, respectively. The only remaining step is to produce the middle vertical arrow. This follows from the following lemma.

Lemma 4.3. *Define*

$$\tilde{\theta}_{n,j} : \mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^nX) \to P\Omega^{n-1}X$$

as follows. Let $(c, \gamma) \in \mathcal{E}_n(j; P\Omega^{n-1}X, \Omega^nX)$ where $c = \langle c_1, \dots, c_j\rangle$ and $\gamma = (\gamma_1, \dots, \gamma_j)$. For $t \in I$ and $v \in S^{n-1}$, define

$$\tilde{\theta}_{n,j}(c, \gamma)(t)(v) = \begin{cases} \gamma_r(s)(u) & if \quad c_r(s, u) = (t, v), \\ \gamma_r(1)(u) & if \quad t \geq c_r'(1), c_r''(u) = v, \gamma_r \notin \Omega^nX, \\ * & else. \end{cases}$$

Then the maps $\tilde{\theta}_{n,j}$ assemble into a map $\tilde{\theta_n}$ such that composition with $E_n\tilde{\eta_n}$ defines the map $\tilde{\alpha_n}$ making the main diagram commute.