Operads and Loop Space Machinery Seminar

Week 3: Exercises

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(1) Show that the category of endofunctors of $\mathcal{C}$ is a monoidal category.
(2) Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction. Show that the composition $G \circ F : \mathcal{C} \to \mathcal{C}$ is a monad in $\mathcal{C}$.
(3) Let $\mathcal{O}$ be the Barratt-Eccles operad from last week, and let C be its associated monad from Construction 3.2. Let $X = S^1$ be the circle. What is $F_n CX$? Compute the mod 2 homology of the filtration quotient F_2CX/F_1CX.
(4) Let $F : \mathcal{C} \to \mathcal{C} : G$ be an adjunction, and let C be a monad in $\mathcal{C}$. Show that FCG is a monad in $\mathcal{C}$.
(5) For Proposition 3.4 of the lecture notes this week, we needed to define a map $\beta_n X : C_n X \to \Omega C_{n-1} \Sigma X$ such that $\alpha_n = (\Omega \alpha_{n-1} \Sigma)\beta_n$. Let $c \in \mathcal{O}_n(j)$, let $x \in X^j$, and let $t \in I$. Write $c_r = c'_r \times c_r" : J \times J^{n-1} \to J^n$. Let $r_1, \ldots, r_i$, in order, denote the indices r such that $t \in c'_r(J)$. Define β_n by

$$[\beta_n(c,x)](t) = \begin{cases} * & if \quad t \notin \cup_{r=1}^{j} c'_r(J), \\ (\langle c_{r_1}", \ldots, c_{r_i}" \rangle, (x_{r_1}, s_1), \ldots, (x_{r_i}, s_i)) & if \quad c'_{r_q}(s_q) = t, 1 \leq q \leq i, \\ & \quad \text{and } t \notin c'_r(J) \text{ for } r \notin \{r_q\}. \end{cases}$$

   (a) Show that β_n is well-defined and continuous.
   (b) Show that $\Omega \alpha_{n-1} \Sigma \circ \beta_n = \alpha_n : C_n X \to \Omega^n \Sigma^n X$.
   (c) Show that β_n is a morphism of monads.
(6) Read Section 7 of May's "The geometry of iterated loop spaces".