Operads and Loop Space Machinery Seminar
Week 4: Minicourse Part IV, The Recognition Principal
J.D. Quigley

1. References.

This part of the minicourse will be based on Sections 9 and 11-14 of May's "The geometry of iterated loop spaces". We are also grateful to Jonathan Rubin and Jens Jakob Kjaer for clarifying some of this material.

2. The two-sided bar construction

In the exercises for the first week, we introduced simplicial objects, and in the case of simplicial sets, we proved that there is an equivalence of categories between pointed simplicial sets and pointed spaces, given by geometric realization and singular chains. Many objects in topology and algebra are naturally simplicial objects; we illustrate this in the following example.

Example 2.1. *Let R be a commutative unital ring, and let M be an R-module. Define the* bar construction *of M to be the simplicial R-module $B_\bullet(R, R, M)$ defined by setting*

$$B_q(R, R, M) = R \otimes R^{\otimes q} \otimes M,$$

with simplicial structure maps $\partial_i : R^{\otimes q+1} \otimes M \to R^{\otimes q} \otimes M$ given by multiplying the i and $(i+1)$-st factors, or acting on M if $i = q$, and s_i given by inserting 1 in the i-th spot of the tensor product of R's.

We can define a chain complex of R-modules $B_*(R, R, M)$ by setting $d_q : B_q \to B_{q-1}$ to be the alternating sum of the ∂_i's. Then $B_*(R, R, M)$ is a free resolution of M, and we have $B_*(R, R, M) \simeq M$, where here $\simeq$ means quasi-isomorphism.

If we repeat replace the left-hand R above by some other R-module N, we can define the two-sided bar construction *$B_\bullet(N, R, M)$. The homology of the resulting complex satisfies*

$$H(B_*(N, R, M)) \cong Tor_R(N, M).$$

To summarize, in the above example we "resolved" an object M by replacing it with a suitably nice simplicial object. Notice that we only used the monoidal structure of R (i.e. multiplication and unit) and the module structure map on M to give this definition. It turns out that we can give this construction in much more generality. First, we need one definition.

Definition 2.2. Let (C, μ, η) be a monad in $\mathcal{C}$. A *C-functor* (F, λ) in a category $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation $\lambda : FC \to F$ such that $id = \eta \circ (F\eta)$ and $\lambda \circ (F\mu) = \lambda \circ \lambda$.

Example 2.3. (1) *If (C, μ, η) is a monad in $\mathcal{C}$, then (C, μ) is a C-functor in $\mathcal{C}$.*

(2) *If $\alpha : C \to D$ is a morphism of monads, then any D-functor can be regarded as a C-functor by pullback, and in the case $C = C_n$ and $D = \Omega^n\Sigma^n$ discussed last time, we get a morphism of C_n-functors*

$$\alpha : C_n \to \Omega^n\Sigma^n.$$

We can now define the monadic bar construction.

Construction 2.4. *Let (C, μ, η) be a monad in $\mathcal{C}$, let (F, λ) be a C-functor in $\mathcal{D}$, and let (X, ξ) be a C-algebra. Define the* monadic bar construction $B_\bullet(F, C, X)$ *as the simplicial object in $\mathcal{C}$ with*

$$B_q(F, C, X) = FC^qX,$$

with face and degeneracy operators given by

$$\partial_0 = \lambda, \quad \lambda : FC^qX \to FC^{q-1}X,$$
$$\partial_i = FC^{i-1}\mu, \quad \mu : C^{q-i+1}X \to C^{q-i}X, \quad 0 < i < q,$$
$$\partial_q = FC^{q-1}\xi, \quad \xi : CX \to X,$$
$$s_i = FC^i\eta, \quad \eta : C^{q-i}X \to C^{q-i+1}X, \quad 0 \leq i \leq q.$$

This is a resolution of a C-algebra X in the same sense that the two-sided bar construction was a resolution of an R-module M in our beginning example. In fact, it is also a resolution of a C-functor F in this sense:

Proposition 2.5. (1) *Let (C, μ, η) be a monad in $\mathcal{C}$ and let (X, ξ) be a C-algebra. Then $\epsilon_\bullet(\xi) : B_\bullet(C, C, X) \to X_\bullet$ is a morphism of simplicial C-algebras in $\mathcal{C}$ and $\tau_\bullet(\eta) : X_\bullet \to B_\bullet(C, C, X)$ of simplicial objects in $\mathcal{C}$ such that $\epsilon_\bullet(\xi) \circ \tau_\bullet(\eta) = 1$ on $X_\bullet$. Moreover, $X_\bullet$ is a strong deformation retract of $B_\bullet(C, C, X)$ in $s\mathcal{C}$.*

(2) *Similarly, if (F, λ) is a C-functor in $\mathcal{D}$ and $Y \in \mathcal{C}$, then $F_\bullet Y_\bullet$ is a strong deformation retract of $B_\bullet(F, C, CY)$ in $s\mathcal{D}$.*

Proof. The explicit simplicial homotopies are written in Propositions 9.8 and 9.9 of May's "Geometry of iterated loop spaces". □

3. Simplicial version of the Recognition Principle

Recall we are trying to prove the Recognition Principle:

Theorem 3.1 (Recognition Principle)**.** *There exists Σ-free operads $\mathcal{O}_n$, $1 \leq n \leq \infty$, such that every n-fold loop space is a $\mathcal{O}_n$-space and every connected $\mathcal{O}_n$-space has the weak homotopy type of an n-fold loop space.*

In this section, we obtain the simplicial version of this theorem. We can apply this to the above machinery to particular case where $C = C_n$ is the monad from the Approximation Theorem and F is either C_n viewed as a C_n-functor, or F is $\Omega^n\Sigma^n$. Combined with the morphism of C_n-functors induced by the morphism of monads $\alpha_n : C_n \to \Omega^n\Sigma^n$ defined last week, we obtain for any C_n-algebra $X \in Top_*$,

$$X_\bullet \xleftarrow{\epsilon_\bullet(\xi)} B_\bullet(C_n, C_n, X) \xrightarrow{B_\bullet(\alpha_n, 1, 1)} B_\bullet(\Omega^n\Sigma^n, C_n, X),$$

where the first map is a strong deformation retract so it has a right-inverse, and the second map is an equivalence since $C_n \simeq \Omega^n\Sigma^n$ as monads and therefore as C_n-functors by the Approximation Theorem.

We can extend this diagram further using the following lemma.

Lemma 3.2. *Let (F, λ) be a C-functor in $\mathcal{D}$ and let $G : \mathcal{D} \to \mathcal{D}'$ be any functor. Then $(GF, G\lambda)$ is a C-functor in $\mathcal{D}'$, and*

$$B_\bullet(GF, C, X) = G_\bullet B_\bullet(F, C, X)$$

in $s\mathcal{D}'$ for any C-algebra X. Here, $G_\bullet$ is defined by applying G to each level of the simplicial object $B_\bullet(F, C, X)$.

Applied in our setting with $G = \Omega^n : Top_* \to Top_*$, we obtain an equivalence

$$B_\bullet(\Omega^n\Sigma^n, C_n, X) = \Omega^n_\bullet B_\bullet(\Sigma^n, C_n, X)$$

where here Σ^n is though of as a C_n-functor in a way we will describe later. Composing our equivalences, we obtain for any C_n-algebra X an equivalence of simplicial spaces

$$X_\bullet \simeq \Omega^n_\bullet B_\bullet(\Sigma^n, C_n, X).$$

4. Geometric realization and proof of the Recognition Principle

We now wish to take the simplicial statement above and promote it to a space-level statement using geometric realization. In the exercises from the first week, we studied some properties of the geometric realization functor from simplicial spaces to spaces. We can use Exercise 4 from there to show the following.

Lemma 4.1. *Suppose X is simply connected or an H-space. Then $|X_\bullet| \simeq X$.*

Therefore we understand the geometric realization of the left-hand side above. For the other side, we use the following result. Note that if $\mathcal{O}$ is any operad and C is its associated monad in $\mathcal{C}$, then there is a natural homeomorphism $\nu : |C_\bullet X| \to C(|X|)$ for $X \in s\mathcal{C}$. Moreover, this extends to a functor from the category of simplicial C-algebras in $\mathcal{C}$ to the category of C-algebras in $\mathcal{C}$.

Theorem 4.2. *For $X \in sTop$, $|P_\bullet X|$ is contractible and there are natural maps $\tilde{\gamma}$ and γ such that the following diagram commutes:*

$$\begin{array}{ccccc} |\Omega_\bullet X| & \longrightarrow & |P_\bullet X| & \xrightarrow{p_\bullet} & |X| \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle =} \\ \Omega|X| & \longrightarrow & P|X| & \xrightarrow{p} & |X| \end{array}$$

Here, $P_q X = PX_q$. Moreover, if X is proper and each X_q is connected, then $|p_\bullet|$ is a quasi-fibration with fiber $|\Omega_\bullet X|$ and therefore $\gamma : |\Omega_\bullet X| \to \Omega|X|$ is a weak homotopy equivalence.

We will construct this diagram and show that the middle terms are contractible, but we will leave the verification that the top row is a quasi-fibration when X is proper and each X_q is connected to the exercises.

Proof. First, note that the standard contracting homotopy on PY is natural in Y for any $Y \in Top_*$. Therefore it commutes with simplicial structure maps to give a simplicial contracting homotopy

$$I_\bullet \times P_\bullet X \to P_\bullet X.$$

Therefore $|P_\bullet X$ is contractible by Exercise 4 of Week 1.

To construct the map $\tilde{\gamma} : |P_\bullet X| \to P|X|$, let $f \in PX_q$, $u \in Delta^q$ and let $t \in I$. Define

$$\tilde{\gamma}|f, u|(t) = |f(t), u|.$$

This is well-defined and continuous, and restricts to an inclusion γ. □

The last step in the proof of the Recognition Principle is to show that n-fold loops is a morphism of C_n-algebras.

Theorem 4.3. *For $X \in sTop_*$, the iteration $\gamma^n : |\Omega^n_\bullet X| \to \Omega^n|X|$ of γ is a morphism of C_n-algebras, and the following diagram commutes:*

$$\begin{array}{ccc} |(C_n)_\bullet X| & \xrightarrow{\nu} & C_n\,|X| \\ \Big\downarrow{\scriptstyle |(\alpha_n)_\bullet|} & & \Big\downarrow{\scriptstyle \alpha_n} \\ |\Omega^n_\bullet \Sigma^n_\bullet X| & \xrightarrow{\Omega^n\tau^n\gamma^n} & \Omega^n\Sigma^n|X| \end{array}$$

Proof. The proof relies on the geometry of the little n-cubes operad and will be left for the exercises. □

Therefore the equivalence from the end of the previous section becomes

$$X \simeq \Omega^n|B_\bullet(\Sigma^n, C_n, X)|,$$

which exhibits X as an n-fold loop space.

Corollary 4.4. *Suppose X is an algebra over an E_n-operad $\mathcal{O}$ with associated monad C. Then X is equivalent to an n-fold loop space.*

Proof. This follows from the equivalence

$$B_\bullet(C_n, C, X) \simeq B_\bullet(\Omega^n\Sigma^n, C, X)$$

where the C-functor structures come via pullback along the equivalence of operads $\alpha : \mathcal{O}_n \to \mathcal{O}$. □