ON DL - operations

Let X be a 1-conn E_1-space, then we have maps $\Sigma_{2+} \wedge X^{\wedge 2} \longrightarrow X$, equiv so in particular $X^{\wedge 2} \longrightarrow X$

Let $H_* = H_*(\ , \mathbb{F}_2)$. Then $\tilde{H}_*(X \wedge X) = \tilde{H}_*(X) \otimes \tilde{H}_*(X)$

So we get a graded mult on $H_*(X)$. $*$ called the pontryagin product.

If X is E_2 then $*$ is graded commutative.

Assume X is E_∞ and 1-conn, ex. $X \simeq \Omega^\infty \Sigma^\infty Y$, Y 1-conn

then we have maps: $E\Sigma_{k+} \wedge X^{\wedge k} \longrightarrow X$ Σ_k-equiv for all k.

so we have maps $E\Sigma_{k+} \wedge_{\Sigma_k} X^{\wedge k} \longrightarrow X$.

We want to compute $\tilde{H}_*(E\Sigma_{k+} \wedge_{\Sigma_k} X^{\wedge k})$.

This can be done by $C_*^{cell}(E\Sigma_{k+}) \otimes_{\Sigma_k} C_*(X)^{\otimes k} \otimes \mathbb{Z}/2$ by Kunneth iso.

Let us pick $k=2$. Then $E\Sigma_{k+} = \cancel{\mathbb{RP}^\infty}\ S^\infty_+$, so $C_*(E\Sigma_{k+})$ is a ~~free~~ resolution of $\mathbb{Z}$ as a triv Σ_{2+}-module

$$\mathbb{Z} \leftarrow \mathbb{Z}[\Sigma_2] \xleftarrow{\sigma - 1} \mathbb{Z}[\Sigma_2] \xleftarrow{1+\sigma} \mathbb{Z}[\Sigma_2] \xleftarrow{\sigma - 1} \mathbb{Z}[\Sigma_2] \leftarrow \dots$$ σ gen Σ_2.

So $C_*(E\Sigma_{2+}) \simeq \mathbb{Z}[\ \ \oplus \mathbb{Z}[\sigma]$

Chain htpy Σ_2-equiv

So $C_*(E\Sigma_{n+})\otimes \mathbb{Z}/2 \simeq ($

$$\mathbb{F}_2\otimes\mathbb{F}_2 \xleftarrow{\epsilon} \mathbb{F}_2[\Sigma_2] \xleftarrow{\sigma+1} \mathbb{F}_2[\Sigma_2] \xleftarrow{\sigma-1} \cdots$$

Let $e_i = 1+\sigma \in \mathbb{F}_2[\Sigma_2]$ in deg i, e_0 = the non basept.

Note $d(e_i)=0$, and $\sigma(e_i)=e_i$

So $e_i\otimes x^{\otimes p} \in C_*(E\Sigma_{n+})\otimes_{\Sigma_p} C_*(X)^{\otimes p}\otimes \mathbb{Z}/2$ is a cycle for

x a cycle in $C_*(X)\otimes\mathbb{Z}/2$ i.e. a homology class in $H_*(X)$.

So $[e_i\otimes x^{\otimes 2}] =: Q^{i+|x|}(x) \in H_*(E\Sigma_{2+}\wedge_{\Sigma_2} X^{\wedge 2})$. So using the

structure map gives ~~$Q^{i+|x|}(x) \in H_*(X)$~~ $Q^i: H_*(X)\to H_{*+i}(X)$

where $Q^i x = 0$ if $i<|x|$.

Note: $Q^{|x|}x = x * x$, by construction Compare w. Steenrod operations.

How do we get Q^iQ^j? Want relations

$$\begin{tikzcd}
E\Sigma_{2+}\wedge_{\Sigma_2}(E\Sigma_{2+}\wedge_{\Sigma_2} X^{\wedge 2})^{\wedge 2} \arrow[r] \arrow[d, "\text{exercise}"'] & E\Sigma_{2+}\wedge_{\Sigma_2} X^{\wedge 2} \arrow[r] & X \\
E\Sigma_{2+}\wedge E\Sigma_{2+}^{\wedge 2}\wedge_{\Sigma_2\wr\Sigma_2} X^{\wedge 4} \arrow[r] & E\Sigma_{4+}\wedge_{\Sigma_4} X^{\wedge 4} \arrow[ur, "\text{structure map}"'] &
\end{tikzcd}$$

$\Sigma_2\wr\Sigma_2 < \Sigma_4$

applying chains as before gives

$$e_i \otimes (e_j \otimes x^{\otimes 2})^{\otimes 2}$$

$$\downarrow$$

$$e_i \otimes e_j^{\otimes 2} \otimes x^{\otimes 4} \longrightarrow \boxed{} \otimes x^{\otimes 4}$$

So we need to study the cells of $E\Sigma_4$, and how it relates to $E\Sigma_2 \times E\Sigma_2^{\times 2}$.

Thm: $Q^i Q^j x = \sum_k \binom{k+j-i}{2k-i} Q^{i+j-k} Q^k$

What if X was only E_{n+1}?

We still have $Q^i : H_s(X) \to H_s(X)$ for $i-s<n$

by comparing $C_n(k)$ with Σ_n.

Further we have $\lambda_n : \tilde{H}_i(X) \otimes \tilde{H}_j(X) \to H_{i+j+n} X$

with if X is C_{n+2} then $\lambda_n = 0$, $\lambda_0(x,y) = [x,y] \leftarrow$ commutator

and some other properties.

The general story line.

Note homology suffices is fine

~~Def~~ Def: X is H_∞-spectrum if it has structure maps

$$E\Sigma_{k+} \wedge_{\Sigma_k} X^{\wedge k} \longrightarrow X \text{ for all } k,$$ satisfying compatibility

in the htpy cat.

~~Let E be a structured~~

Ex: X is an E_∞ space $\Rightarrow \Sigma^\infty X$ is H_∞.

E is a structured ring spectrum (multiplicative coh-thy)

Then $\alpha \in E_* X \leftrightsquigarrow S^i \xrightarrow{\alpha} E \wedge X \overset{\text{Free E-mod map.}}{\rightsquigarrow} E \wedge S^i \xrightarrow{\alpha} E \wedge X$

apply $E\Sigma_{k+} \wedge_E (\ )^{\wedge_E k} \rightsquigarrow E \wedge S^{k\cdot i}_{h\Sigma_k} \longrightarrow E \wedge X^{\wedge p}_{h\Sigma_p} \longrightarrow E \wedge X$

We name this composite $P(\alpha)$ total power operation on α.

Apply π_* gives $E_* S^{k\cdot i}_{h\Sigma_k} \longrightarrow E_* X$

Note: $E = H\mathbb{F}_p$, $k=2$ then $S^{2\cdot i}_{h\Sigma_2} \overset{\text{Exercise}}{=} \Sigma^i(\mathbb{RP}^\infty/\mathbb{RP}^{i-1})$

$$H_*(\Sigma^i(\mathbb{RP}^\infty/\mathbb{RP}^{i-1})) = \mathbb{F}_2\{Q^i, Q^{i+1}, \dots\} \quad |Q^j| = i+j$$

then $P(\alpha)(Q^j) = Q^j\alpha$ from before.

So every interesting class in $E_*(S^{k\cdot i}_{h\Sigma_k})$ $k>1$ gives a homology operation!

Silly example: $E = H\mathbb{Q}$. $H\mathbb{Q}_*(S^{2\cdot i}_{h\Sigma_2}) = \begin{cases} \mathbb{Q} & i \text{ even}, * = 2\cdot i \\ 0 & \text{else} \end{cases}$

The non trivial class is $x \longmapsto x\cdot x$

$$H_*(S^{n\cdot i}_{h\Sigma_n};\mathbb{Q}) = \begin{cases} 0 & i \text{ is odd} \\ \mathbb{Q} & i \text{ is even, and } ~~\text{concentrated in deg}~~ * = n\cdot i. \end{cases}$$

and the action is $\alpha \longmapsto \overbrace{\alpha\cdot\ldots\cdot\alpha}^{n}$.

Note that X fin CW cmplx in spaces $\mathbb{S}^{X_+}$ is E_∞ so H_∞

$E_*\,\mathbb{S}^X = E^*X$ and D-L operations become coh operations

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How do we get relations?

$$\begin{tikzcd}
E\wedge (S^{ni}_{h\Sigma_n})^{\wedge l}_{h\Sigma_l} \arrow[r] \arrow[d] & E\wedge (X^{\wedge n}_{h\Sigma_n})^{\wedge l}_{h\Sigma_l} \arrow[d] \arrow[dr] & \\
E\wedge S^{n\cdot l\cdot i}_{h\Sigma_l\wr\Sigma_n} \arrow[r] \arrow[d] & E\wedge X^{\wedge n\wedge l}_{h\Sigma_l\wr\Sigma_n} \arrow[d] & E\wedge X^{\wedge l}_{h\Sigma_l} \arrow[d] \\
E\wedge S^{n\cdot l\cdot i}_{h\Sigma_{n\cdot l}} \arrow[r] \arrow[u, bend left] & E\wedge X^{n\cdot l}_{h\Sigma_{n\cdot l}} \arrow[r] & E\wedge X
\end{tikzcd}$$

This is induced by $\Sigma_l\wr\Sigma_n \subset \Sigma_{n\cdot l}$, so we need to study E_*-grp homology.