E_∞ - rings

Recall a naive spectrum X is a collection of ptd spaces X_i, w. $\Sigma X_i \to X_{i+1}$.

Given $X \in Top_*$ we obviously have $\Sigma^\infty X$ in an obvious way.

One would want a sym mon prod $\wedge$ on Sp, s.t $\Sigma^\infty X \wedge \Sigma^\infty Y \overset{\simeq}{\leftrightarrow} \Sigma^\infty X \wedge Y$.

Unfortunately this is not possible. So we will equip our sp w. more structure:

Def: [May] A orthogonal spectrum X is a collection of spaces, for $V \subseteq \mathcal{U} \cong \mathbb{R}^\infty$ real inner product space, X_V, w. a $O(V)$-action, s.t if $V \subset W$ then
$$S^{V^\perp} \wedge X_V \to X_W \text{ is } O(V^\perp) \times O(V) \subset O(W)\text{ - equiv.}$$

Ex: • $\mathbb{S}$, $\mathbb{S}_V = S^V$; $\Sigma^\infty X$, $\Sigma^\infty X_V = S^V \wedge X$.

Note: This depends on a choice of $\mathcal{U} \cong \mathbb{R}^\infty$, but any linear isometry $\mathcal{U} \cong \mathcal{U}'$ induces equiv of cat.

Note: X O-sp, we can recover a naive spectrum by $X_n = X_{\mathbb{R}^n}$, and forgetting the action. This induces an equiv of homotopy Cats.

Def: X, Y $\mathcal{U}$-O-sp, then $X \wedge Y$ a $\mathcal{U} \times \mathcal{U} (\cong \mathbb{R}^\infty)$ spectrum by
$$(X \wedge Y)_{V \times W} = X_V \wedge Y_W$$

If $\varphi: \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ linear isometry then we can define a $\mathcal{U}$-spectrum $(X \wedge Y)^\varphi$ by $\quad (X \wedge Y)^\varphi_V = (X \wedge Y)_{\varphi^{-1}(V)}$

Def: $\mathcal{L}_n \cancel{(\mathcal{U}^{\times n}, \mathcal{U})}$ = Lin isometries $\mathcal{U}^{\times n} \to \mathcal{U}$

Note: $\mathcal{L}_2 \simeq *$, actually $\forall n \neq 0$ $\mathcal{L}_n \simeq *$

Further by composition $\mathcal{L}_n$ forms an operad.

So we now have that $\mathcal{O}$-spectra are E_∞-monoidal

— Aside

In fact we can do better by letting

$$(X \wedge Y)_{\mathbb{R}^n} = \mathrm{Coeq}\left(\left(\bigvee_{p+1+q=n} \mathcal{O}(n)_+ \wedge_{\mathcal{O}(p)\times 1 \times \mathcal{O}(q)} X_{\mathbb{R}^p} \wedge S^1 \wedge Y_{\mathbb{R}^q} \right) \rightrightarrows \left(\bigvee_{p+q=n} \mathcal{O}(n)_+ \wedge_{\mathcal{O}(p)\times \mathcal{O}(q)} X_{\mathbb{R}^p} \wedge Y_{\mathbb{R}^q} \right) \right)$$

This is sym-monoidal.

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Def: E is an E_∞-ring Sp if it is a monoid w.r.t to this one.

equivalently, we have compatible maps $\varphi \in \mathcal{L}_n$

$$E_{V_1} \wedge \dots \wedge E_{V_n} \to E_{\varphi(V_1, \dots, V_n)}$$

Note: If X is a spectrum then $\Omega^\infty X$ is a E_∞-space. If E is E_∞-ring spectrum then $\Omega^\infty E$ is E_∞ in two different ways! How do they interact?

Here is something iffy going on, because for $\mathcal{O}$-Sp our spheres S^V come w. $\mathcal{O}(V)$-action. So we need our delooping theory to be okay w. this

For any inner product space V, we can define C_V little cubes operad but it does not have a $\mathcal{O}(V)$ action, as it could deform cubes.

So what about D_V the little disc-operad? It does have a $O(V)$ action, and following what we did in JD's mini course we can show that X is a $\overset{\text{comm}}{}D_V$-space (w. the action) $\iff \exists Y$ s.t. $X \simeq \Omega^V Y$ ($O(V)$-space)

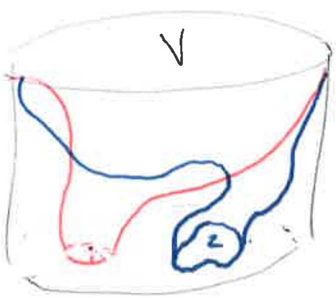
The problem is that we need maps that deals w. suspension, and D_V does not do this! Since $D^V \times D^W \neq D^{V+W}$ ...

i.e. if $V \subset W$, $\Omega^V \Omega^{V^\perp} X \simeq \Omega^W X$ there is no maps of operads $D_V \to D_W$ witnessing this!

So we need to build a better operad!

Def Let V be inner prod space. $E_V = \{\varphi : V \to V \mid$ distance shrinking embedding $\}$

$\mathcal{K}_V^{(n)} = \{(\alpha_1, \dots, \alpha_n) \mid \alpha_i : I \to E_V,\ \alpha_i(1) = id,\ \alpha_i(0)(V) \cap \alpha_j(0)(V) = \emptyset$ for $i \neq j\}$



$V \times I$

$\in \mathcal{K}_V(2)$

Prop: ① $\mathcal{K}_V$ forms an operad by composition in E_V.

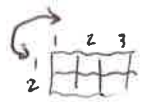
② $\mathcal{K}_V \to D_V$ is an equivalence of operads, proj on to $\alpha(0)$.

③ $V \subset W \rightsquigarrow \mathcal{K}_V \hookrightarrow \mathcal{K}_W$ compatible w. $\Omega^V \Omega^{V^\perp} X \simeq \Omega^W X$.

④ $\mathcal{K} := \operatorname{Colim}_{V < \mathcal{U}} \mathcal{K}_V$ is E_∞.

~~Def~~: Returning to ~~[illegible]~~ $\Omega^\infty E$

Def: $(\mathcal{G},\mathcal{C})$ is an operad pair if, $\mathcal{G}$ and $\mathcal{C}$ are operads w. structure maps $\mathcal{G}(k)\times \mathcal{C}(j_1)\times\dots\times\mathcal{C}(j_k)\to\mathcal{C}(j_1\cdots j_k)$ $\quad \Sigma_k\times\Sigma_{j_1}\times\dots\times\Sigma_{j_k}<\Sigma_{j_1\cdots j_k}$ equiv.

Note. $\Sigma_2\times\Sigma_3\times\Sigma_2\to\Sigma_6$ as 

- $\mathcal{G}$ multiplies the addition $\mathcal{C}$

Ex: (Com, Com) because everything is basept.

$(\mathcal{L},\mathcal{K})$

only known examples!

Ex: $(\mathcal{L},\mathcal{K})$, $\varphi\in\mathcal{L}_k$

$$\mathcal{K}_{V_1}(j_1)\times\dots\times\mathcal{K}_{V_k}(j_k)\longrightarrow\mathcal{K}_{\varphi(V_1\times\dots\times V_k)}(j_1\cdots j_k)$$

and then let $(V_1,\dots,V_k)$ range over $\mathcal{U}\times\dots\times\mathcal{U}$.

Def: • X is a $(\mathcal{G},\mathcal{C})$-space if it is a $\mathcal{G}$- and $\mathcal{C}$-space s.t. $CX\to X$ map of $\mathcal{G}$-algs, (Note CX being $\mathcal{G}$-space is non-trivial)

- $(\mathcal{G},\mathcal{C})$ is E_∞-pair if $\mathcal{G}$ and $\mathcal{C}$ are E_∞
- X is an E_∞-ring space if it is a $(\mathcal{G},\mathcal{C})$-space for an E_∞-pair.

Result: E E_∞-ring spectrum $\Rightarrow$ $\Omega^\infty E$ is a $(\mathcal{L}, \mathcal{K})$-space

Result: if $(\mathcal{G}, \mathcal{C})$ is an E_∞-pair, and X is a $(\mathcal{G}, \mathcal{C})$-space, then we can deloop w.r.t to $\mathcal{C}$ to get an E_∞-ring spectrum.

Note: We could also deloop w.r.t $\mathcal{G}$, but that wrecks the $\mathcal{C}$ structure simply giving us a spectrum.

Note: E_∞-ring spaces in nature rarely occurs from an operadic pt of view, but rather either as Thom spectra or bipermutative categories, or deloopings structured ring sp.

If X is a $(\mathcal{G}, \mathcal{C})$ E_∞-rings space then as it is E_∞ from $\mathcal{C}$ we have DL operations Q^i, and pontryagin product $*$ on $H_*(X; \mathbb{F}_2)$ as it is E_∞ from $\mathcal{G}$ we get DL-operations $\bar{Q}^i$ and pontryagin product $\circledast$.

Prop: $x, y, z \in H_* X$ $\quad \Delta x = \Sigma x' \otimes x''$

(1) $(x * y) \circledast z = \Sigma (-1) (x \circledast z') * (y \circledast z'')$

(2) $\bar{Q}^s(x * y) = \sum\limits_{s_0+s_1+s_2} \Sigma \Sigma \bar{Q}^{s_0}(x') * \bar{Q}^{s_1}(x'' * y') * \bar{Q}^{s_2}(y'')$

(3) $\bar{Q}^r Q^s(x) = \sum\limits_{i,j,k} \Sigma \binom{r-i-2k-1}{s+j-i-k} Q^i \bar{Q}^j(x') * Q^{r+s-i-j-k} \bar{Q}^k x''$