Geometric Group Theory

Week 1

1 What is geometric group theory?

Geometric group theory is a collection of tools used to study algebraic properties of groups by examining the spaces on which they act. It is an extremely wide area of mathematics, pulling concepts from all over geometry, topology, and algebra. The essence of geometric group theory is best understood with a sample tool:

Proposition 1. If a group G acts freely by isometries on $\mathbb{R}^n$, then G is torsion-free.

Proof. Suppose that $g \in G$ has finite order m. Choose $v \in \mathbb{R}^n$, and consider the orbit $\mathcal{O}$ of v under g:

$$\mathcal{O} = \{v, g \cdot v, g^2 \cdot v, \ldots, g^{m-1} \cdot v\}.$$

Let $w \in \mathbb{R}^n$ be the centroid of $\mathcal{O}$ (Potential Exercise: prove every finite collection of points in $\mathbb{R}^n$ has a centroid). Then, we have

$$g \cdot \mathcal{O} = \{g \cdot v, g^2 \cdot v, \ldots, g^m \cdot v = v\} = \mathcal{O}.$$

This, along with the fact that G acts by isometries, implies that $g \cdot w = w$. Since the action is free, it must be that $g = id$. So, we have shown that the only finite order element of G is the identity, and so G is torsion-free. □

In this example, the algebraic property we were concerned about was torsion. However, there are a myriad of other properties one may be interested in. Some of these properties include:

- Torsion
- Free
- Finiteness properties
  - Finitely generated/presented
  - Cohomological dimension
  - Type F, FP, FL, FH, FA

- "Virtual" properties

For the rest of today, we will talk about a particular one of these examples – namely, free groups. Let's start with the definition:

Definition 1. The *free group on n generators* $x_1, x_2, \ldots, x_n$, denoted F_n consists of all finite words of the form

$$s_1^{k_1} s_2^{k_2} \cdots s_m^{k_m}$$

where $s_1, \ldots, s_m \in \{x_1, \ldots, x_m\}$ and $k_1, \ldots, k_m \in \mathbb{Z} \setminus \{0\}$. The group operation is concatenation of words.

Free groups are perhaps the most fundamental groups since every group is a quotient of a free group (this is essentially what we mean when we write the presentation of a group). However, it is in general difficult to tell when a group we encounter in the wild is free. Here's an example that hopefully convinces the reader:

Example 1. Let $m \geq 3$ and consider the projection $p : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/m\mathbb{Z})$ given by reducing each entry modulo m. The kernel of this projection is called the *level m congruence subgroup* of $\mathrm{SL}(2, \mathbb{Z})$, and is denoted $\mathrm{SL}(2, \mathbb{Z})[m]$. Perhaps surprisingly, it turns out that $\mathrm{SL}(2, \mathbb{Z})[m]$ is free. By the end of the lecture, we should have the tools necessary to justify this.

So, it may be useful to have some tools we can use to tell whether or not a given group is free. The rest of the lecture will be dedicated to constructing these tools.

2 Tools

Proposition 2 (Ping-Pong Lemma)**.** Suppose a group G is generated by the set $\{g_1, \ldots, g_n\}$ and acts on a set X. If

(a) there are nonempty, pairwise-disjoint subsets $X_1, \ldots, X_n \subseteq X$, and

(b) $g_i^k(X_j) \subseteq X_i$ for all $k \neq 0$ and $i \neq j$,

then G is a free group of rank n.

Proof. The idea will be to show that all nontrivial freely reduced words in the g_i's represent nontrivial elements of G. This is sufficient because then G cannot possibly have any relations (i.e. G is *free* is relations). Let w be a nontrivial freely reduced word in the g_i's. We will consider two cases:

- Case 1: Suppose w is of the form $g_1^* s g_1^*$, where the stars represent a nontrivial power and s is some word in the g_i's. Then, for any $x_2 \in X_2$, we have $w \cdot x_2 \in X_1$ by property (b). Since X_1 and X_2 are disjoint, we have $w \cdot x_2 \neq x_2$, and so w must be nontrivial.

- Case 2: Suppose the w is not of the form $g_1^* s g_1^*$. Then w is conjugate to an element of this form (conjugate by g_1), which is nontrivial by Case 1. Since w is conjugate to a nontrivial element, it must be nontrivial itself.

$\square$

Applying the Ping-Pong Lemma can be tricky (you have to find a suitable set X and subsets $X_1, \dots, X_n$), but we will give an example of such an application.

Example 2. Consider the subgroup

$$A = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{SL}(2, \mathbb{Z})$$

which acts on $\mathbb{Z}^2$ by matrix multiplication. Let

$$X_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 : |x| > |y| \right\} \qquad \text{and} \qquad X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 : |x| < |y| \right\}.$$

Then, for $k \neq 0$ and $\begin{pmatrix} x \\ y \end{pmatrix} \in X_2$, we have

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ky \\ y \end{pmatrix}.$$

Since $k \neq 0$ and $|x| < |y|$, it follows that $|y| < |x + 2ky|$. So, $\begin{pmatrix} x + 2ky \\ y \end{pmatrix} \in X_1$. Almost exactly the same argument shows that

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^k \begin{pmatrix} x \\ y \end{pmatrix} \in X_2$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in X_1$. Therefore, by the Ping-Pong Lemma, $\left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle$ is a free group of rank 2.

The Ping-Pong Lemma has a couple limitations from our perspective. The first, that we've noted already, is that finding good choices for X can be difficult. The second is that the Ping-Pong Lemma doesn't really utilize any geometric information of the action; X is simply a set. So, if we let G act on a *space*, we should be able to get some stronger (and more easily applicable!) results. The simplest spaces on which groups can act are graphs, and particularly trees. So what can be said regarding the free groups and the action of groups on trees? We have the following classical theorem in geometric group theory:

Theorem 1. *If a group G acts freely on a tree T, then G is a free group.*

Proof. Our method is going to be to find nice *tiling* of T. By a *tile*, we mean a subtree of the barycentric subdivision T' of T, and by a *tiling* of T, we mean a collection of tiles such that

- No two tiles share an edge, so two tiles can only intersect in at most one vertex of T'.
- The union of the tiles is all of T'.
- There is a tile T_0 such that every tile is of the form $g \cdot T_0$ for some $g \in G$.

The experienced reader will recognize T_0 as a "fundamental domain" for the action of G on T'.

- **Step 1: Tile the tree T.**
  Fix a vertex $v \in T$. For $g \in G$, let T_g be the subtree of T' spanned by vertices $w \in T'$ satisfying $d(x, gv) \leq d(w, g'v)$ for all $g' \in G$. Exercise: Show that $\{T_g\}$ is a good tiling.

- **Step 2: Find a generating set.**
  Let $S = \{g \in G : (gT_{id}) \cap T_{id} \neq \emptyset\}$. We will show that S generates G. Let $g \in G$. Since T is connected, there is a path from v to gv. Suppose this path goes through the tiles $T_{g_0}, T_{g_1}, \ldots, T_{g_m}$ (in that order), where $g_0 = id$ and $g_m = g$. Let $s_i = g_{i-1}^{-1} g_i$. Since the path goes from $T_{g_{i-1}}$ to T_{g_i}, $T_{g_{i-1}} \cap T_{g_i} = \{*\}$. Hitting both sides with g_{i-1}^{-1}, we get $T_{id} \cap T_{g_{i-1}^{-1} g_i} = \{*\}$. Therefore, $s_i = g_{i-1}^{-1} g_i \in S$. Finally, we notice that $g = s_1 s_2 \cdots s_m$, which completes the proof.

- **Step 3: Show S is a free generating set for G.**
  Let $g = s_1 s_2 \cdots s_m$, where $s_i \in S$. We can find a path in T' from v to gv passing through the tiles

  $$T_{id}, \quad T_{s_1}, \quad T_{s_1 s_2}, \quad T_{s_1 s_2 s_3}, \quad \ldots, \quad T_{s_1 s_2 \cdot s_m} = T_g.$$

  Moreover, this path is unique since T' is a tree. Therefore, the decomposition $g = s_1 s_2 \cdots s_m$ is also unique since a different decomposition would give a different path from v to gv. Thus, S is a free generating set for G.

□

The ideas used in this proof are used frequently throughout geometric group theory: connectedness of the space allows you to find a path and generating set, and uniqueness of the path allows you to say something about uniqueness of the decomposition.

It turns out that the converse of this theorem is also true; that is, every free group acts freely on a tree. This follows because the Cayley graph of a free group is a tree (we will discuss this next week). Therefore, this we have found a complete classification of free groups!

Finally, we note that, if a group acts freely on a tree, then any subgroup also acts freely on that tree. Therefore, we get the following corollary:

Corollary 1. Every subgroup of a free group is free.

This may seem obvious, but it is actually quite nontrivial to prove using purely algebraic or combinatorial methods. This is just one example of geometric methods giving nontrivial results in algebra.

Intrestingly enough, the rank of a free group does not respect subgroups. For instance, the subgroup

$$\langle bab^{-1}, b^2ab^{-2}, b^3ab^{-3}, \ldots \rangle \subseteq F_2$$

is a free group of infinite rank.