[redacted] ...

(1) <u>Def</u>: A <u>left ordering</u> on a gp G is a total ordering $<$ s.t. $\boxed{g<h \Rightarrow fg<fh}$ $\forall f,g,h\in G$.

"$<$ is left-invariant"

Dually, a <u>right ordering</u> satisfies $g<h \Rightarrow gf<hf$

A <u>bi-ordering</u> is an ordering which is both left & right invariant.

<u>Exercise</u>: (i) left order $\Leftrightarrow$ $P\subset G$ s.t. $PP\subseteq P$ & $G = P\sqcup\{1\}\sqcup P^{-1}$ (positive elts), (ii) G left orderable $\Leftrightarrow$ G right orderable (iii) biorderings $\Leftrightarrow$ $\exists$, P is conj-invariant

<u>Obs</u>: A left-orderable group is torsion-free

$g^n=1 \leadsto$ wlog $g<1 \Rightarrow 1=g^n<g^{n-1}<\dots<g<1$

<u>Conj (Kaplansky)</u>: G torsionfree gp, R: domain $\Rightarrow$ $R[G]$: domain

known for orderable groups

<u>Examples</u>:

- An abelian group is left-orderable iff it is biorderable, iff it is torsion-free (obvious in f.g. case) - (Levi)
- F_n is biorderable. More generally, the free product of biorderable groups is biorderable.
- thm of Dehornoy $\Rightarrow$ B_n is left orderable, but not biorderable for $n\geq 3$. PB_n is biorderable. More generally, the semidirect product of biorderable groups is biorderable.

(We will not prove all these facts)

- More generally, $MCG(\Sigma,\partial\Sigma)$ is left-orderable if Σ is an n-punctured orientable surface w/ nonempty boundary.

(2)

~~Obs: Suppose G~~

~~$\langle a \rangle_{x>}$~~

Obs: • Suppose that $G \to \mathrm{Homeo}_+(\mathbb{R})$

& $r \in \mathbb{R}$ has ~~...~~ trivial stabilizer.

Then define $g > h \Leftrightarrow g(r) > h(r)$. This is a left ordering: $fg > fh \Leftrightarrow f(g(r)) > f(h(r))$ b/c $f \in \mathrm{Homeo}_+(\mathbb{R})$.

& if $g(r) = h(r)$, then $g^{-1}h \in \mathrm{Stab}(r)$

$\Rightarrow g^{-1}h = 1$

$\Rightarrow g = h$.

• Suppose that $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$ is injective

& G is countable.

Then G is left-orderable.

Pf: Let $Q \subseteq \mathbb{R}$ be a ctble dense subset & well-order Q. Define $f < g$ if

$q_* = \min \{q \mid f(q) \neq g(q)\} \rightsquigarrow f(q) < g(q)$

"alphabetical wrt Q"

This is a left ordering: left inv ✓

& if $f(q) = g(q)$ $\forall q$ then $f = g$ by density.

Thm (Ghys): The converse also holds

(if ctble G is left ordarable, then $G \hookrightarrow \mathrm{Homeo}_+(\mathbb{R})$)

③' "The" ordering on B_n, or more generally on $MCG(\Sigma, \partial\Sigma)$

Note: $\underbrace{MCG(D^2_n, \partial D^2)}_{G} \xrightarrow{\sim} B_n$

· Construction of map:

$\varphi \in G \Rightarrow \varphi$ can be "filled in" to $\Phi \in MCG(D^2, \partial D^2)$

Lemma: $MCG(D^2, \partial D^2) = \text{triv}$.

Pf: $\Psi \rightsquigarrow$ Let $\Psi_t = \begin{cases} \text{rescaled } \Psi \text{ on } D^2(t) \\ \text{id on } D^2 \setminus D(t) \end{cases}$

↳ homotopes to identity

So Φ is up to isotopy ~~[illegible]~~ the identity.
Choose an isotopy to the identity.
The n filled-in punctures trace out a braid.
Isotopy of Φ's yields isotopy of braids.
Clearly a homomorphism.

- Surjectivity: easy to see that gen's are hit.

· Injectivity: $MCG(D^2_n, \partial D^n \cup n\text{ pts}) = \text{triv}$.

Pf is similar to lemma

(4) **Fact:** Let $(\Sigma, \partial\Sigma)$ be a surface w/ bdy & punctures s.t. $\chi(\Sigma) < 0$.

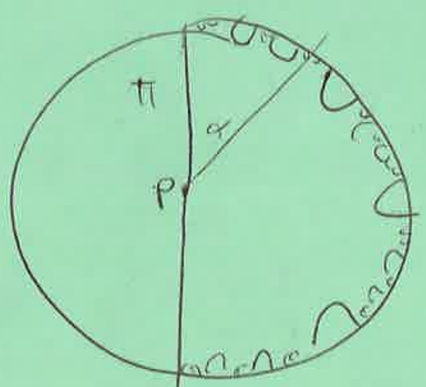
Then Σ admits a hyperbolic metric such that

- each puncture is a cusp (lies @ ∞ distance from pts in Σ)
- each bdy component is a geodesic.

Cor: The universal cover $\tilde{\Sigma}$ of Σ embeds in $\mathbb{H}^2$ in such a way that

- each puncture is on the circle @ ∞.
- $\tilde{\Sigma}$ is geodesically convex

Define an action of $MCG(\Sigma, \partial\Sigma) \curvearrowright \mathbb{R}$ as follows.



- Pick a base pt $p \in \partial\tilde{\Sigma}$, let π be the component of $\partial\tilde{\Sigma}$ containing p.
- $\partial(\overline{\Sigma}) \approx S^1$, so $\partial(\overline{\Sigma}) \setminus \overline{\pi} \approx \mathbb{R}$
- By geodesic convexity, every pt in $\partial(\overline{\Sigma}) \setminus \overline{\pi}$ admits a unique geodesic to p.

  ↳ So $\partial(\overline{\Sigma}) \setminus \overline{\pi}$ is parameterized by the angle α that this geodesic makes w/ π.

- $\varphi \in MCG(\Sigma, \partial\Sigma)$ lifts to a unique map $(\tilde{\Sigma}, \partial\tilde{\Sigma}) \to (\tilde{\Sigma}, \partial\tilde{\Sigma})$ fixing p.

  ↳ This extends to a unique map $(\overline{\Sigma}, \cdot) \to (\overline{\Sigma}, \cdot)$.

  Since π is fixed, so is $\overline{\pi}$, & so this restricts to a map $\partial(\overline{\Sigma}) \setminus \overline{\pi} \to \partial(\overline{\Sigma}) \setminus \overline{\pi}$.

(5') This map is isotopy-invariant: if $f \sim g \in MCG(\Sigma, \partial\Sigma)$, then f & g induce the same map on $\pi_1(\Sigma, p)$

A pt on $\partial\tilde{\Sigma}$ is det'd by a pt in $\partial\Sigma$ & $\alpha \in \pi_1(\Sigma, \partial\Sigma)$, so f & g induce the same map on $\partial\tilde{\Sigma}$. By density, they induce the same map on $\overline{\partial\tilde{\Sigma}}$.

It is clearly a homomorphism, so we get a map $MCG(\Sigma, \partial\Sigma) \to \mathrm{Homeo}_+(\mathbb{R})$.

<u>Claim:</u> this map is injective.

<u>Pf:</u> Suppose that $f \in MCG(\Sigma, \partial\Sigma)$ has $\bar{f}$ fixing $\mathrm{Homeo}_+(\mathbb{R})$. Then $\pi_1(f)$ is trivial.

But Σ is aspherical
$\Rightarrow f$ is htic to the identity.

This implies f is isotopic to the identity. (standard theorem).

<u>Cor:</u> $MCG(\Sigma, \partial\Sigma)$ is left-orderable.

(6') There is a simpler way to define an order from this construction:

Let $s_1 \dots s_k$ be gen'rs of $\pi_1(\Sigma)$
w/ $p_1 \dots p_k$ lifts of endpts to $\tilde{\Sigma}$
& define $\varphi > 1$ if $\bar{\varphi}(p_1) > p_1$

or $\bar{\varphi}(p_1) = p_1$ & $\bar{\varphi}(p_2) > p_2$

or - - -

or $\bar{\varphi}(p_1) = p_1$ & $\dots$ & $\bar{\varphi}(p_{k-1}) = \bar{\varphi}(p_{k-1})$ & $\bar{\varphi}(p_k) > \bar{\varphi}(p_k)$

Total order b/c if $\pi_1(\varphi) = 1$, then $\varphi = 1$ in $MCG(\Sigma, \partial\Sigma)$

~~Rf That are left orderable gp inject into order pres permutations of an ordered set.~~

Note: There is an explicit ordering of B_n.

Define a braid word w to be positive if for the lowest i s.t. $\sigma_i^{\pm 1}$ occurs in w, it occurs only positively (as σ_i^{+1} rather than σ_i^{-1})

Thm: Define a braid to be positive if it can be rep'd by a positive braid word. This is a left ordering.

Note: B_n is not biorderable $(n \geq 3)$. If it were, there would be an ordering invariant under inner auts. But conj by the cycling aut Δ_n sends $\sigma_i \mapsto \sigma_{i+1}$ cyclically $\Rightarrow\Leftarrow$.