Ordered groups exercises

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Definition 1. *Define a* positive cone *in a group G to be a subset $P \subset G$ such that*

1. *$G = P \amalg P^{-1} \amalg \{1\}$, i.e. P is disjoint from $P^{-1} = \{p^{-1} \mid p \in P\}$ and every nonidentity element of G is in either P or P^{-1}.*
2. *$PP \subseteq P$ where $PP = \{pq \mid p, q \in P\}$.*

Say that a positive cone $P \subset G$ is normal *if $P = gPg^{-1} = \{gpg^{-1} \mid p \in P\}$ for each $g \in G$.*

Exercise 1. *Let G be a group*

1. *Show that there is a bijection between positive cones on G and left orderings on P.*
2. *Deduce by duality that a G is left orderable if and only if it is right orderable.*
3. *Show that the bijection of (1) restricts to a bijection between normal positive cones on P and bi-orderings of P.*

Exercise 2. Kaplansky's conjecture *asserts that if k is a field (or more generally a domain) and G is a torsion-free group, then the group ring $k[G]$ has no nonzero zero divisors.*

1. *Show that if G has torsion, then $k[G]$ has nonzero zero divisors for every nonzero ring k.*
2. *Show that if G is left-orderable and k is a (possibly noncommutative) ring without nonzero zero divisors, then $k[G]$ has no nonzero zero divisors.*

Exercise 3.

1. *Show that a group G is left orderable if and only if it acts faithfully on a linearly ordered set via order-isomorphisms.*
2. *Show that every countable linear order embeds in $\mathbb{Q}$.*
3. *Show that a countable group G is left orderable if and only if it acts faithfully on $\mathbb{Q}$ via order-isomorphisms, if and only if it acts faithfully on $\mathbb{R}$ via order-isomorphisms.*

Exercise 4. *Let* $(\Sigma, \partial\Sigma)$ *be a surface with nonempty boundary and* n *punctures. Show that* $\mathrm{Mod}(\Sigma, \partial\Sigma)$ *acts faithfully on* $\pi_1(\Sigma)$.

Exercise 5. *Show that there is no bi-ordering on the braid group* B_n.