The Mostow Rigidity Theorem

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(1) Hyperbolic Geom

Def A Riem. mfld (M^n, d) is hyperbolic if it has constant negative curvature

Ex $\mathbb{D}^n$, $\mathbb{H}^n$ – isometric models for Hyperbolic n-space

$\mathbb{D}^n$ = open n-disk in $\mathbb{R}^n$, with metric

$$d = \frac{d_{Euc}}{\text{scale factor}},$$ factor $\longrightarrow \infty$ as you move towards boundary

$$\mathbb{H}^n = \{(x_1, x_2, \ldots x_n) \mid x_1 > 0\} \subseteq \mathbb{R}^n$$

with $$d = \frac{d_{Euc}}{x_n^2}$$

fact If M conn/simply conn, complete, hyperbolic

$$M \cong_{isometric} \mathbb{D}^n$$

Idea: Fix p, look at $\exp : T_pM \longrightarrow M$
Then if M non positive curvature (not nec constant)
this is surj by complete and inj by s.c. so a diffeo
Then, we add that the only s.c. constant neg curvature
mfld (up to isometry) is $\mathbb{D}^n$

Cor If M is a conn complete, then

$$\tilde{M} \simeq E(\pi_1 M), \quad M \simeq B(\pi_1 M)$$

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(univ cover)

Rmks about diffeo, non-isometric structures on a mfld
e.g. $\mathbb{D}^n_{Euc} \cong \mathbb{D}^n_{hyp}$

<u>Thm</u> (Mostow Rigidity)

Suppose M, N hyperbolic mflds, compact, dim ≥ 3 and $f: M \to N$ is a htpy equiv.
Then f is htpic to an isometry, which is unique

<u>Cor</u> M, N as above. If $\pi_1 M \cong \pi_1 N$, then M & N are isometric

<u>Pf of Cor</u> For any G, m, if X_1 & X_2 are both models for $K(G,m)$
then $\exists$ htpy equiv $X_1 \to X_2$

<u>Proof outline</u>

Let $f: M \to N$ htpy equiv of cpt hyperbolic mflds

(1) Lift f to $\tilde{f}: \mathbb{D}^n \to \mathbb{D}^n$ (universal covers)

(2) Show that $\breve{f}$ is a quasi-isometry

Apply the idea that our q-isom looks nicer & nicer as distances $\to \infty$

(3) Show $\bar{f}$ is a quasi-conformal map on $\partial \mathbb{D}^n$

(4) Show $\bar{f}$ is, in fact strict conformal

(5) Show a conformal map $\partial \mathbb{D}^n \to \partial \mathbb{D}^n$ extends to isometry $\mathbb{D}^n \to \mathbb{D}^n$

Let $f: M \to N$ be a htyp equiv. $g: N \to M$ htpy inverse

Lift to $\tilde{f}: \underset{\mathbb{D}^n}{\overset{\cong}{\tilde{M}}} \to \underset{\mathbb{D}^n}{\overset{\cong}{\tilde{N}}}$, $\tilde{g}$

WLOG, all maps are smooth

Claim $\tilde{f}$ is a quasi-isometry, in particular $\exists\, a \geq 1,\ b \geq 0$ s.t. $\forall m_0, m_1 \in \tilde{M}$

$$\frac{1}{a} d(m_0, m_1) - b \leq d(\tilde{f}(m_0), \tilde{f}(m_1)) \leq a\, d(m_0, m_1) + b$$

Pf $|df|$ is bdd since f is smooth btwn cpt mflds

thus $|d\tilde{f}|$ is bdd, similarly for g (since the covering map is locally an isometry)

thus $\exists\, c$:

$$d(\tilde{f}(m_0), \tilde{f}(m_1)) \leq c\, d(m_0, m_1)$$

$$d(\tilde{g}(n_0), \tilde{g}(n_1)) \leq c \cdot d(n_0, n_1)$$

We can take a smooth htpy $F: M\times I \longrightarrow M$, $id_M \sim g\circ f$
& lift to
$\tilde{F}: \tilde{M}\times I \longrightarrow \tilde{M}$

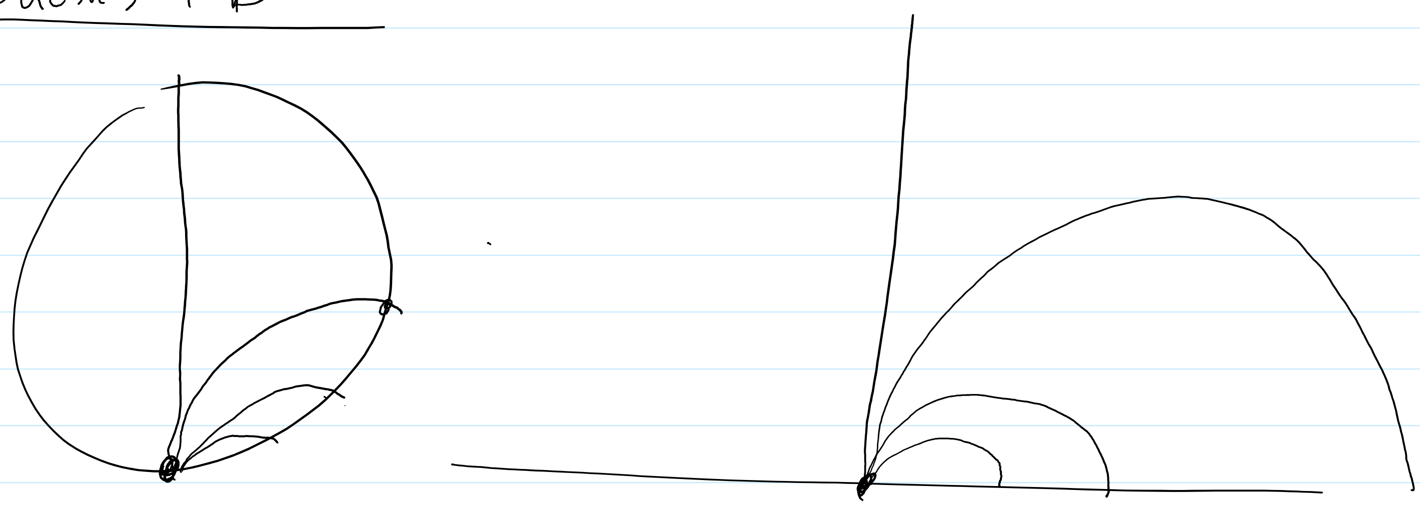
& same argument shows $\exists\, c'$

$$d\big(F(m_0,1),\ F(m_0,0)\big) \leq c'$$
$$\| $$
$$d\big(\hat{g}(\hat{f}(m_0)),\ m_0\big)$$

Combine to get our desired (a couple steps, but not enlightening)

Leverage this weak type of map by looking out to ∞

Geodesics in $\mathbb{D}^n$



Let $\partial\mathbb{D}^n$ & $\partial\mathbb{H}^n$ denote the S^{n-1} at ∞ - are Riemannian mflds w/ Euc metric (we lose the "stretched" hyperbolic direction)

$\overline{\mathbb{D}^n} = \mathbb{D}^n \cup \partial\mathbb{D}^n$ & $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$

"ends"

$$\{\text{geodesics}\} \underset{\longleftarrow}{\xrightarrow{\text{"ends"}}} \{\text{pairs in } \partial\}$$

& the isom $\mathbb{D}^n \longrightarrow \mathbb{H}^n$ preserves ends

ends can be done intrinsically using geodesic rays

Next: extend a quasi-isometry $D \longrightarrow D$ to the boundary.

Let $\phi : D \longrightarrow D$ q.isom.

Prop if $\gamma \subseteq D$ is a geod. then $\exists! \ \tilde{\gamma} \subseteq D$ s.t.

$$\operatorname{dist}(\tilde{\gamma}, \phi(\gamma)) < \infty$$

pf let $[x,y]$ = segment, $\ell(x,y)$ = geod, N_t = t-ball

Parametrize γ by arc len, so $\gamma \cong \mathbb{R}$

Claim $\exists \ t$ s.t. $\forall \ x,y \in \gamma$,

$$\phi([x,y]) \subseteq N_t(\ell(\phi(x), \phi(y)))$$

Suppose not:

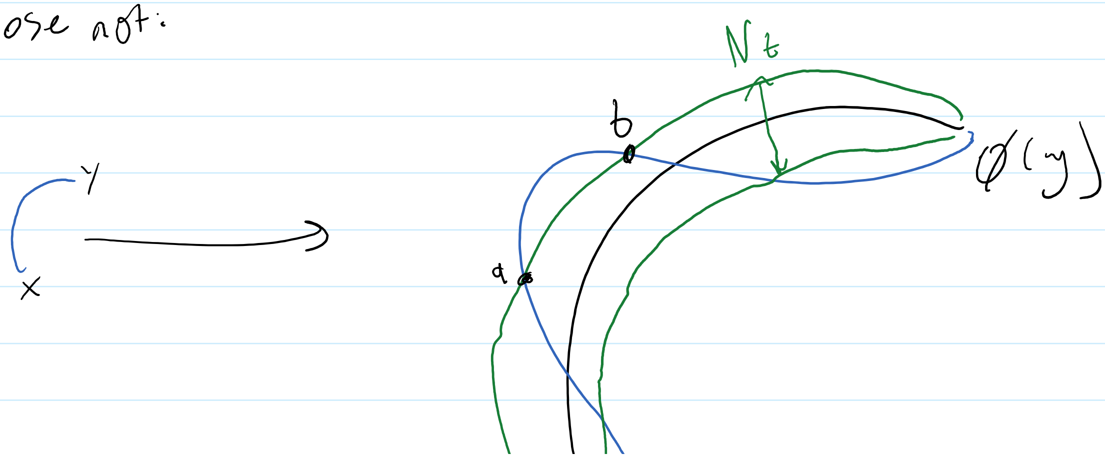
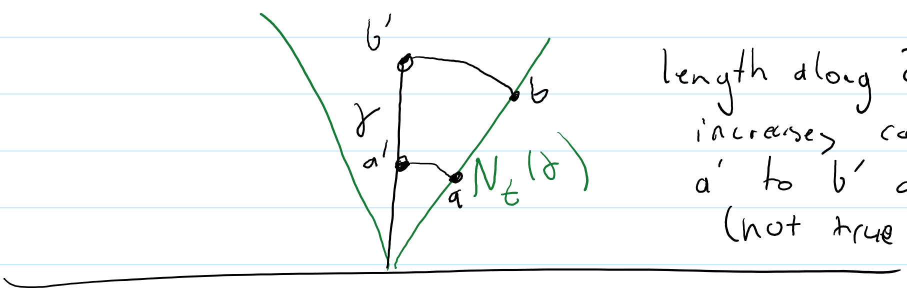


Image of $[x,y]$ has to leave & re-enter N_t, so pick a & b as shown

Then $\phi(\gamma)$ has this outside segment, whose length has a lower bound depending on t, but whose length has an upper bound by q.isom

$\Longrightarrow$ such a t exists

Here, negative curvature is essential: we need length along the boundary of N_t to scale up as t increases. this can be seen in half-plane model:



length along ∂N_t from a to b increases compared to along γ from a' to b' as a function of t (not true in euclidean)

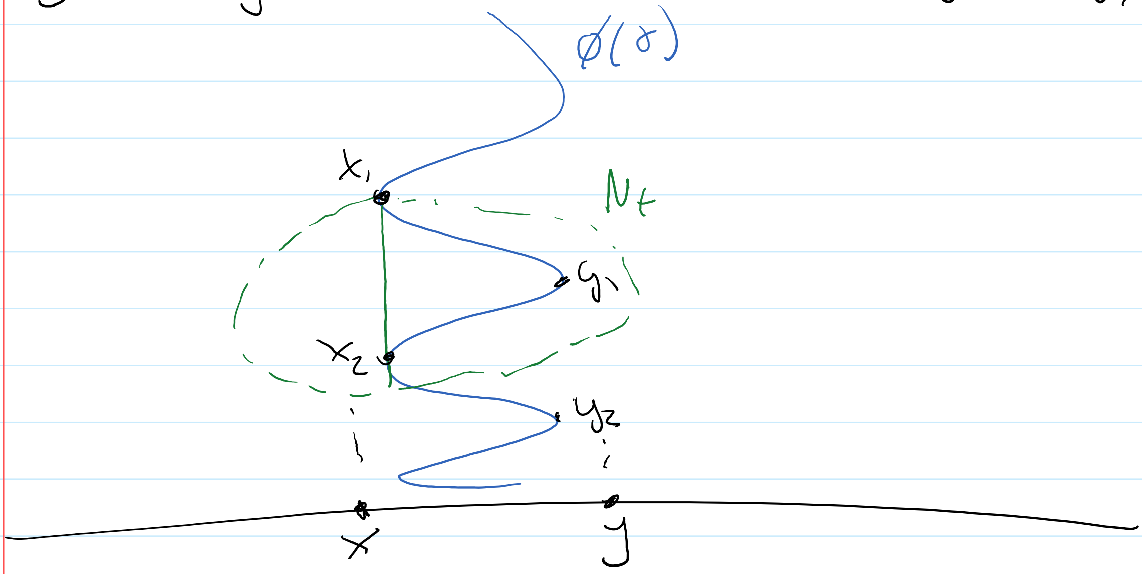
Now, no seq $x_i \in \gamma$, $x_i \to \infty$ can have

$\phi(x_i)$ converge on interior of D by q isom lower bound

So cluster pts of Φ are on ∂D

Say x_i $u_i \to \infty$, $\phi(x_i) \to x$ $\phi(u_i) \to u$

Say $x_i, y_j \to \infty$, $\phi(x_i) \to x$, $\phi(y_i) \to y$,



By claim, $N_t(\ell(\phi(x_1), \phi(x_n))$

contains $y_1, \ldots y_m$, where $m \to \infty$ as $n \to \infty$

So these seqs are within t of eachother,
but, if $x \neq y$, then they are ∞ apart

So $x = y$

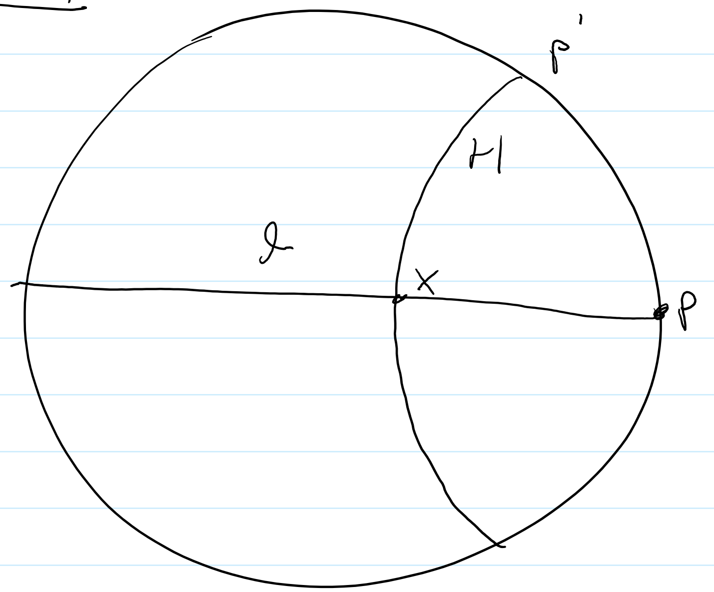
Then set $\tilde{\gamma} = \ell\left(\lim_{x \to \infty} \phi(x), \lim_{t \to \infty} \phi(t)\right)$.

Def $\bar{\phi} : \bar{D} \to \bar{D}$:

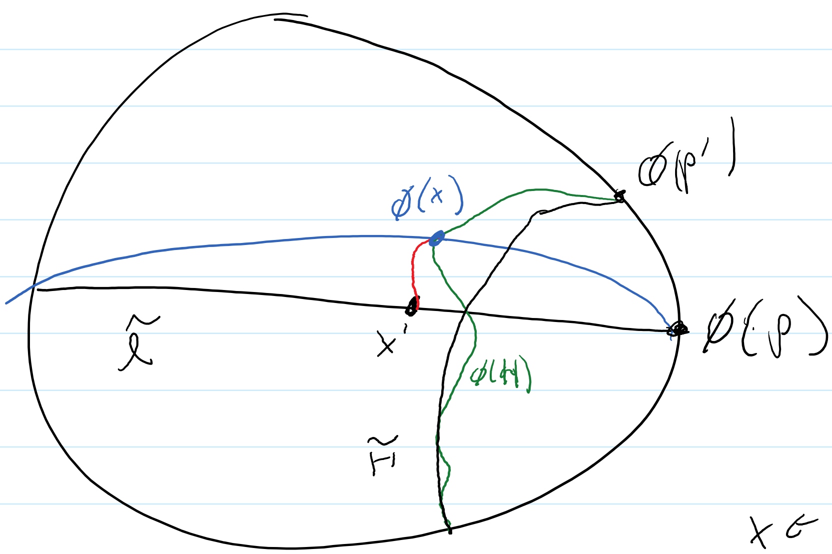
if $x \in \partial D$, pick any γ with end x,
& let $\bar{\phi}(F) =$ end of $\tilde{\gamma}$

Next we show $\bar{\phi}$ is continuous (w/r/t usual Euclidean metric on the boundary)

Idea:



Look at geodesic through p' $\perp$ to one through p



$x \in H, l \mid \phi(x) \in \tilde{H}, \tilde{l}$

$\phi(H)$ near $\tilde{H}$ + $\phi(x)$ near x'
$\Rightarrow \phi(H)$ near x'

As $x \longrightarrow p$, x' and $\phi(x) \longrightarrow \phi(p)$

As $p' \longrightarrow p$, $x \longrightarrow p$, so $\phi(p') \longrightarrow \phi(p)$ ✓

Now return to $\phi = \tilde{f}$, our homotopy equiv.

and we show $\tilde{f}\big|_{\partial D}$ is a homeomorphism.

The htpies which exhibit $\tilde{f}, \tilde{g} : D \longrightarrow D$ as inverses extend to

$$\bar{D} \times I \longrightarrow \bar{D}$$

but since I is compact, it can't effect distances too much, thus it's restriction to ∂D must be the identity, so they are honest inverses.

For R. Mflds D, D', and

$$\varphi : D \longrightarrow D$$

a homeo, we set

$$L, \ell \quad D \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}$$

$$L(P, r) = \sup_{d(x,P)=r} d(\varphi(x), \varphi(P))$$

$$\ell(P, r) = \inf_{\quad ''} \qquad ''$$

$$H(P) = \limsup_{r \to 0} \frac{L(P,r)}{\ell(P,r)}$$

<u>Def</u> φ is <u>quasi-conformal</u> if H is bounded.
K-q. conf if $H \leq K$ a.e.

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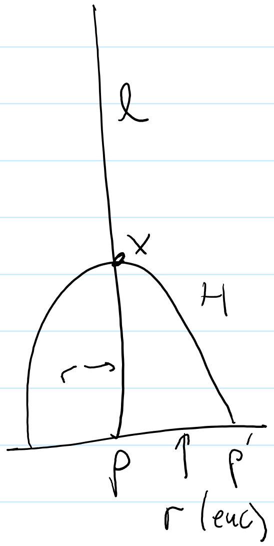
Idea: a conformal map preserves angles. It's allowed to stretch, so long as it stretches equally in all directions.

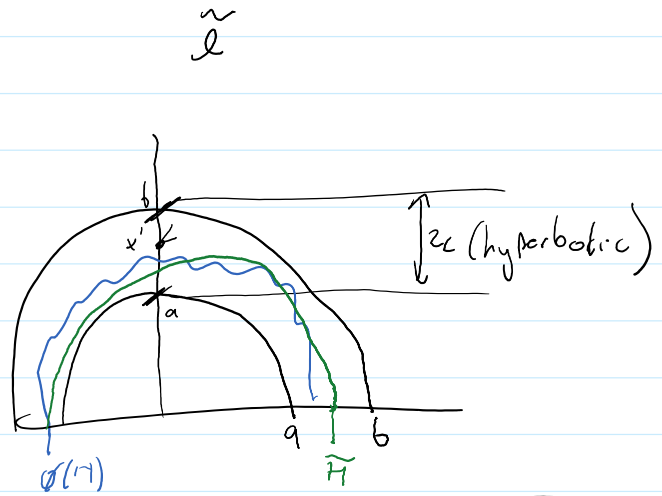
Then a quasi-conformal map stretches differently in different directions, but not too differently

Thm $\bar{f}|_{\partial D}$ is q-conf (∂D has the euclidean metric)

Pf Let $p \in \partial D$, we'll show q conf at p

Easiest to argue in $\mathbb{H}^n$ model





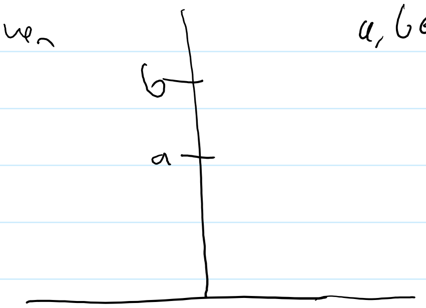
From before: there is a bound c s.t. $\forall$ geodesics J $\perp$ to ℓ through x,

$d(\phi(x), x') \leq c$

$d(\phi(J), \bar{J}) \leq C$

Thus $\phi(J)$ is contained in a shell as shown w/ hyperbolic length along $\hat{l} \leq 2c$

Fact Given $a, b \in \mathbb{R}$, hyp. len$(\overline{ab}) = 2c$,



then $b/a = e^{2c}$

Thus,
$$H(p) \leq e^{2c}$$

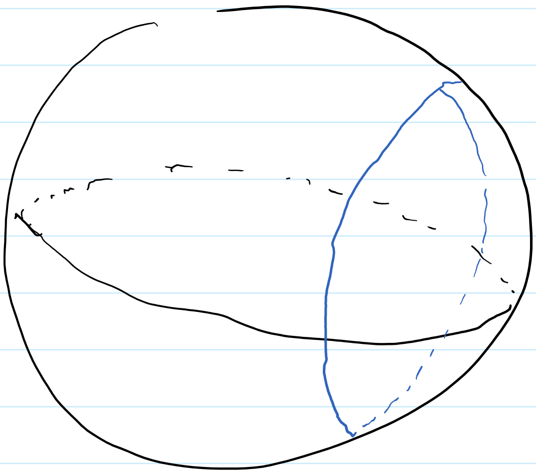
From now on, we're going to hide a lot of technical geometry/analysis details.

Fact Let $\phi: S^{n-1} \longrightarrow S^{n-1}$ be quasi-conformal and equivariant with respect to a cocompact subgroup of $\mathrm{Isom}(\mathbb{D}^n)$

Then ϕ is conformal.

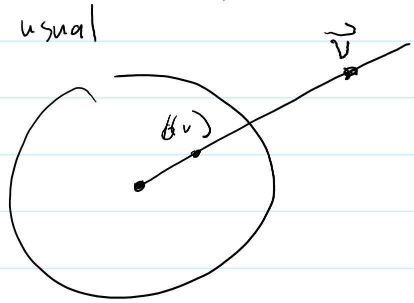
Fact (Liouville's Theorem)

Conformal maps $S^{n-1} \longrightarrow S^{n-1}$ are generated by inversions on circles.



(If we regard S^{n-1} as the one-pt compactification of $\mathbb{R}^{n-1}$, then these inversions are the usual

$$\vec{v} \longmapsto \lambda \frac{\vec{v}}{\|v\|^2}$$



inversions for circles not containing ∞, and reflections in $\mathbb{R}^{n-1}$ for circles containing ∞)

<u>Cor</u> Any conformal map $S^{n-1} \longrightarrow S^{n-1}$ extends to a map

$$\overline{D}^n \longrightarrow \overline{D}^n$$

which is an isometry on the interior w/r/t hyperbolic metric

Now, given our original $\tilde{f}: D \longrightarrow D$, we have the isometry $h: D \longrightarrow D$ given by the cor. Furthermore, $\tilde{f}$ and h have the same extension to the boundary. Thus we can define a homotopy

$$H: D \times I \longrightarrow D$$

by setting $H(x,0) = \tilde{f}(x)$
$H(x,1) = h(x)$

and

$H(x,t) =$ the point along the geodesic $\tilde{f}(x) \rightsquigarrow h(x)$ with the right length.

This homotopy is then equivariant with respect to the same subgroup of isometries that $\tilde{f}$ is, thus it descends to a homotopy from $f: M \longrightarrow N$ to an isometry. $\boxed{\Xi}$