Problem 1. Let G be the group of isometries of $\mathbb{D}^n$, and let Γ, Γ' be isomorphic discrete cocompact subgroups. Show that there exists a $g \in G$ such that $\Gamma = g\Gamma' g^{-1}$. (This is sometimes called the algebraic form of Mostow Rigidity.)

Problem 2. Let $\phi : \mathbb{D}^n \to \mathbb{D}^n$ be a quasi-isometry from the open n-disk to itself in the hyperbolic metric. Recall that we've shown that such a ϕ extends to a function $\overline{\phi} : S^{n-1} \to S^{n-1}$, where for $P \in \partial\mathbb{D}^n$: we define $\overline{\phi}(P)$ by looking at the geodesic which is finite distance the image of the a geodesic ray originating at P.

Show that $\overline{\phi}$ is continuous.

Problem 3. This problem has a bunch of exposition. The actual problems are the three propositions at the end.

Recall that for any two simply connected domains in $\mathbb{C}$, there is a holomorphic map from one to the other with holomorphic inverse. Furthermore, holomorphic maps are conformal (to be defined in a moment). Thus, there are lots of conformal maps in dimension 2. In contrast, in this theorem we'll prove the following:

Theorem 1 (Liouville's Theorem)**.** *Let $f : U \to U'$ be a diffeomorphism between domains in $\mathbb{R}^n$, for $n \geq 3$. If f is conformal, then it is a composition of reflections and inversion through spheres.*

Remark 1. It is not necessary to assume f is smooth with smooth inverse - I believe just f being C^2 is enough - but it simplifies things.

Definition 1. Let $U \subseteq \mathbb{R}^n$ be a domain (an open connected subset) and let $f : U \to \mathbb{R}^m$ be a smooth map which is a diffeomorphism onto its image. Then f is *conformal* if for all $p \in U$, there is a constant $\alpha(p) \in \mathbb{R}_{>0}$ depending smoothly on p such that for all $v, w \in T_pU$,

$$\langle df_p(v), df_p(w)\rangle = \alpha(p)^2 \langle v, w\rangle.$$

Intuitively, the independence of $\alpha(p)$ on v and w means f preserves angles. Here we equip $\mathbb{R}^n$ with the Euclidean metric. The squared is there to compare with $df_p(v) = \alpha(p)v$. As function $U \to \mathbb{R}$, α can't be just anything. We take the following lemma without proof.

Lemma 1. *Let f be conformal, and $\alpha : U \to \mathbb{R}$ as above. If α isn't constant, then for each $p \in U$, the Hessian matrix*

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{\alpha}\right)\right)_{1 \leq i,j \leq n}$$

is diagonal at each point $p \in U$.

Remark 2. This lemma is where dimension at least 3 is used. The Hessian is a symmetric bilinear form defined by the function, and the argument considers a certain alternating form $\Lambda^3(T_p) \to \mathbb{R}$, which needs that $\dim T_p \geq 3$ to be nontrivial.

By taking antideriviatves, we find find:

Corollary 1. *There exist* $x_0 \in \mathbb{R}^n$, $A, B \in \mathbb{R}$ *such that for all* $p \in U \subseteq \mathbb{R}^n$.

$$\alpha(p) = \frac{1}{A|p - x_0|^2 + B}.$$

Remark 3. When $n = 2$ and $f : U \to \mathbb{C}$ is a holomorphic function, then $\alpha(z) = |f'(z)|$, which can be many more things than the above.

Proposition 1. *Either* $A = 0$ *or* $B = 0$.

Proposition 2. *If* $A = 0$, *then* f *is a dilation around* x_0 *plus a translation.*

Proposition 3. *If* $B = 0$, *then* f *is an inveresion around* x_0, *then a dilation, then a translation*

Problem 4. Let $\phi : \mathbb{D}^n \to \mathbb{D}^n$ be a quasi-isometry, and $h : \mathbb{D}^n \to \mathbb{D}^n$ be an isometry such that they both have the same extension to the boundary. Show that they are homotopic. Futhermore, show that if ϕ is a lift of a map $f : M \to N$ of compact hyperbolic manifolds, then this homopty can be chosen to descend to a homotopy of maps $M \to N$.