Mapping Class Groups + the Dehn-Nielsen-Baer Thm

12/5/18

Intro - beautiful thm: interplay btwn topology + alg in the mapping class group

Thm (Dehn-Nielsen-Baer)

For S_g surface w/o bndry of genus $g \geq 1$

$$\mathrm{Mod}^{\pm}(S_g) \xrightarrow[\cong]{\sigma} \mathrm{Out}(\pi_1(S_g, p))$$

I. describe obj.
II. map
III. inj
IV. surj

I. Objects Involved

① Last semester, Jake introduced the mapping class group

orient. pres.

$\mathrm{Mod}(S) := \mathrm{Homeo}^{+}(S, \cancel{\partial S}) / \text{isotopy}$

S ~~cpct~~, orientable surface
connected, w/o bndry

c'ls'd ex: ~~$\mathrm{Mod}(D^2) = 1$~~ $\quad \mathrm{Mod}(S^2) = 1$

$\mathrm{Mod}(T^2) \cong SL(2, \mathbb{Z})$

extended mapping class group $\mathrm{Mod}^{\pm}(S) := \mathrm{Homeo}(S) / \text{isotopy}$

S surface w/o bndry

→ $\mathrm{Mod}(S)$ index 2 subgp of $\mathrm{Mod}^{\pm}(S)$

Ex: $\mathrm{Mod}^{\pm}(S^2) \cong \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Mod}^{\pm}(T^2) \cong GL(2, \mathbb{Z})$

② G grp $\rightsquigarrow$ $\mathrm{Aut}(G)$ = automorph of G

$\mathrm{Inn}(G) = \{\phi \in \mathrm{Aut}(G) \mid$ for some $g \in G$ $\phi(x) = g^{-1}xg \;\forall x \in G\}$

Fact: $\mathrm{Inn}(G)$ is normal subgrp of $\mathrm{Aut}(G)$

Defn outer auto. grp $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$

↳ group of automorph, considered up to conjugation

Ex: $\mathrm{Out}(\mathbb{Z}^n) = GL(n, \mathbb{Z})$

Note: ($\pi_1(T^2) \cong \mathbb{Z}^2$)

II. the map

S surface w/o bndry

Take $p \in S$

Want: $\mathrm{Mod}^{\pm}(S) \longrightarrow \mathrm{Out}(\pi_1(S,p))$

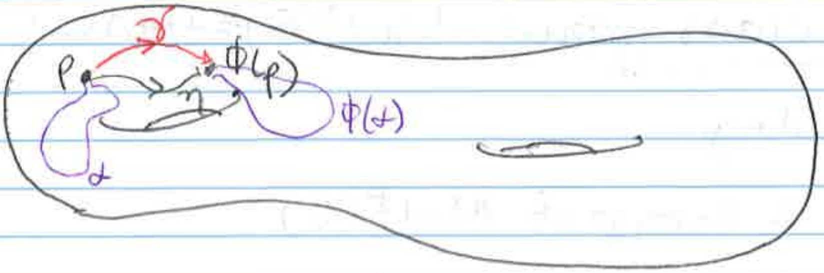
Take homeomorphism $\phi: S \longrightarrow S$

any path $\gamma: p \rightsquigarrow \phi(p)$

$$\begin{tikzcd} \pi_1(S,p) \arrow[rr, dashed, "\phi_*"] \arrow[dr, "\phi_*"'] & & \pi_1(S,p) \\ & \pi_1(S,\phi(p)) \arrow[ur, "\gamma_*"'] & \end{tikzcd}$$

$[\alpha] \mapsto [\phi(\alpha)] \mapsto [\gamma * \phi(\alpha) * \gamma^{-1}]$

ϕ invert ⇓ ϕ_* is ~~iso~~ ~~auto~~ auto morph



Note: - FIX ϕ: diff. choices of γ give diff ϕ_*, but differ by conjugation ($\gamma\eta^{-1}$ on $[\eta * \phi(\alpha) * \eta^{-1}]$.

↳ well-defined elts of $\mathrm{Out}(\pi_1(S))$

- well-defined on $\mathrm{Mod}^{\pm}(S)$:

up to isotopy $\underset{\wedge}{:}$ same htpy class

isotopic maps give same htpy class

↳(htpi'c + ...)

?

Fact: grp. homomorph.

III. Injectivity $S_g, g \geq 1$ $\Rightarrow$ universal cover is contractible

$\Rightarrow$ S_g is a $K(\pi_1(S),1)$-space ?

$\Rightarrow$ have correspondence:

$$\left\{ \begin{array}{l} \text{free htpy classes} \\ \text{of (unbased) maps } S \to S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes} \\ \text{of homomorph} \\ \pi_1(S) \to \pi_1(S) \end{array} \right\}$$

In particular: for identity map $\pi_1(S) \xrightarrow{id} \pi_1(S)$

htp'c to id on S

$\Downarrow$

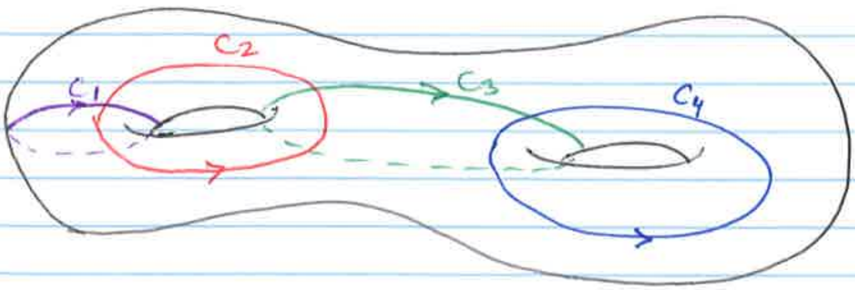
isotopic to id

IV. surjectivity

Take any $[\Phi] \in Out(\pi_1(S_g, p))$

Φ be representative automorphism

- choose chain of isotopy classes of simply closed curves in S_g: $(c_1, \ldots, c_{2g})$



$i(c_j, c_{j+1}) = 1$

$i(c_j, c_k) = 0 \quad |j-k| > 1$

$\hat{i}(c_j, c_{j+1}) = +1$

Aside: geometric intersection # := minimal # of ~~unsigned~~ intersection pts btwn free htpy classes of closed curves a, b

btwn rep of a + rep of b

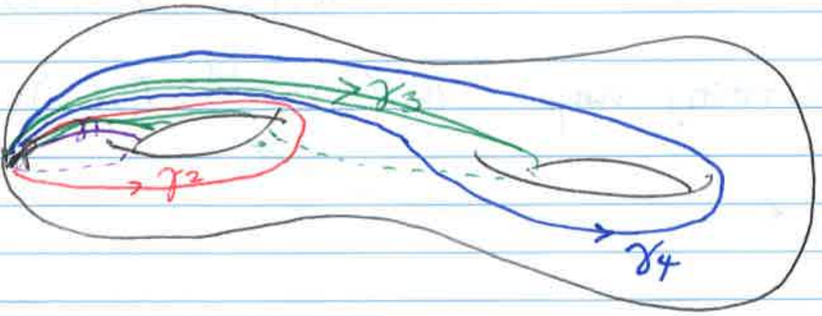
$i(a,b) = \min\{|\alpha \cap \beta| \mid \alpha \in a, \beta \in b\}$

alg int. # = $\hat{i}(\alpha,\beta)$ (α, β transverse, oriented, s.c.c. in S) $\hat{i}(\alpha,\beta)$:= sum of indices of intersection pts of α & β

(+1 if agrees w/ orient of S
−1 else)

Recall: $\left\{\begin{array}{c}\text{free htpy classes}\\ \text{of oriented curves}\\ \text{in } S_g\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{conjugacy classes}\\ \text{of elts in } \pi_1(S_g,p)\end{array}\right\}$

$$\gamma_j \in \pi_1(S_g,p) \longmapsto c_j = \text{conjugacy class of } \gamma_j$$



Φ is auto ~~of~~ $\pi_1(S_g) \Rightarrow$ acts on set of conjg. classes of $\pi_1(S_g)$

Claim: $\{\Phi(c_i)\}$ has intersection patterns we want
↳ also a chain of isotopy classes of s.c.c.

Digression: Intersection stuff

Key-step to pf of DNB Thm: show elt of Out$(\pi_1(S_g))$ (a priori, only preserve ~~top~~ alg info) preserves top'l properties

↳ respect top'l property of whether or not free htpy classes of 2 simply closed curves have intersection # 0

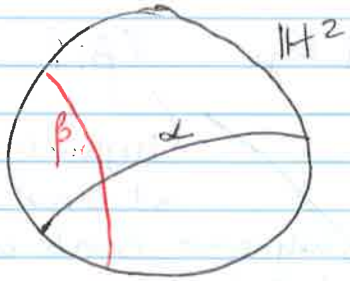
└ do this by studying behavior of $\pi_1(S_g)$ at ∞ in $\mathbb{H}^2$

Let S be hyperbolic surface

Defn $\alpha \in \pi_1(S)$ is hyperbolic if corresponding deck transformation is a hyperbolic isometry of $\mathbb{H}^2$

(Note: if S closed, all non-triv elts of $\pi_1(S)$ are hyperbolic)

Recall: axis of hyperbolic element α of $\pi_1(S)$ has pair of endpts $\partial\alpha$ lying in $\partial\mathbb{H}^2$



Defn 2 hyperbolic elts $\alpha, \beta \in \pi_1(S)$ are linked at ∞ if $\partial\alpha$ separates $\partial\beta$ (in $\partial\mathbb{H}^2$)

Lemma $g \geq 2$. Fix covering map $\mathbb{H}^2 \to S_g$

Let $\Phi \in \mathrm{Aut}(\pi_1(S_g))$

$\gamma, \delta \in \pi_1(S_g)$

$\neq 0$

$\Phi(\gamma)$ & $\Phi(\delta)$ are linked at ∞ iff γ & δ are

pf: - all elts are hyperbolic → statement makes sense

- Φ invert $\Rightarrow$ suffice to show:

If γ, δ not linked $\Rightarrow \Phi(\gamma), \Phi(\delta)$ not linked

Assume γ, δ not have same axis.

Fact: any automorph of fin. gen grp is a quasi-isometry

so $\exists$ $K \geq 1$, $C \geq 0$ s.t. $\boxed{\frac{1}{K} d(\alpha,\beta) - C \leq d(\Phi(\alpha),\Phi(\beta)) \leq K \cdot d(\alpha,\beta) + C}$ ⊛

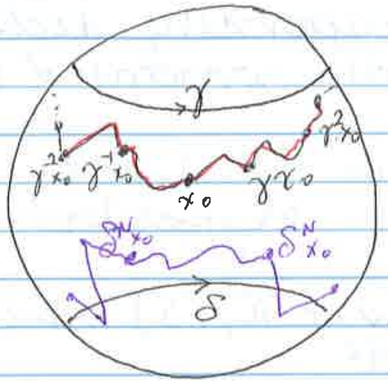
let D = diameter of some fixed fund'l domain for $\pi_1(S_g)$ in $\mathbb{H}^2$

Fix some $R > 2DK^2 + 2CK$

~~Fix some~~ x_0 = bspt in $\mathbb{H}^2$

→ consider orbit $\mathcal{O}_\gamma = \{\gamma^k \cdot x_0 \mid k \in \mathbb{Z}\}$

connect pts by geod. paths, each segment connects 2 pts in adjacent fund'l domains



$\{\alpha_i\}$ — lies in some fixed metric nbhd of axis for γ

→ choose $N = N(R)$ s.t. ($\because \gamma, \delta$ unlinked)

each pt of $\mathcal{O}_{\delta^N} = \{(\delta^N)^k \cdot x_0 \mid k \in \mathbb{Z}, k \neq 0\}$

is distance at least $R+D$ from each pt of $\mathcal{O}_\gamma$ $\{\beta_i\}$ (hyperbolic dist. at least R from $\{\alpha_i\}$)

→ β_i, β_{i+1} also in adjacent fund'l domains

length of any geod. segment $\leq 2D$ ($\because$ in adjacent fundamental domains)

↳ vertices: identified w/ particular elts of $\pi_1(S_g)$

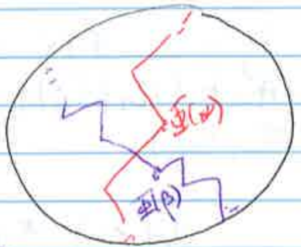
Assume by way of contradiction that $\Phi(\gamma)$ & $\Phi(\delta)$ are linked at ∞

$\Rightarrow$ $\{\Phi(\alpha_i)\}$ & $\{\Phi(\beta_i)\}$ must cross

↳ each geodesic segment must have length

$\leq K(2D) + C$] ⊛

(dist. btwn endpts)



$\Rightarrow$ each segment has at least 1 endpt whose dist from crossing pt $\leq \frac{K(2D)+C}{2}$

$\Downarrow$ dist. btwn endpts is $\leq K(2D)+C$

$\Rightarrow$ so $\exists$ elts $\alpha, \beta \in \pi_1(S_g)$ w/ $d(\alpha,\beta) \geq R$ $(\geq R+D)$

$> 2DK^2 + 2CK$

and $d\left(\Phi(\alpha), \Phi(\beta)\right) \leq K(2D) + C$

$d(\alpha, \beta) \geqslant R > 2DK^2 + 2CK$

And by property of q-isom.

$$\frac{1}{K}(2DK^2 + 2CK) - C < d(\Phi(\alpha), \Phi(\beta)) \left(\leq K(2D) + C \right)$$

$2DK + C$

$\Rightarrow\Leftarrow$

so $\Phi(\gamma), \Phi(\delta)$ must <u>not</u> be linked at ∞.

Fact: 2 conjugacy classes a, b have $i(a,b) = 1$
iff for some $\alpha \in a$ that is linked at ∞ w/ a
given rep $\beta \in b$, set of rep of a that are
linked with β is precisely

$$\{\beta^k \alpha \beta^{-k} \mid k \in \mathbb{Z}\}$$

[10 min.]

(?)

Fact:

→ Change of coord princ: any 2 collections of s.c.c. in S with the same intersection pattern can be taken to each other via an orientation preserving homeo of S) Δ

⟹ ∃ homeo ϕ that fixes basept p & sat.

$$\phi_*(c_i) = \Phi_*(c_i)$$ [action on conjugacy classes]

To finish, wts: $$\underset{\mathrm{Mod}^{\pm}(S)}{\underset{\uparrow}{[\phi]}} \longmapsto \underset{\mathrm{Out}(\pi_1(S_g,p))}{\underset{\uparrow}{[\Phi]}}$$

B/C representatives γ_i generate $\pi_1(S_g,p)$, suffices to show ∃ inner auto $\gamma \xmapsto{I_\alpha} \alpha\gamma\alpha^{-1}$ s.t. $I_\alpha \circ \phi_*^{-1} \circ \Phi(\gamma_i) = \gamma_i$

$\alpha \in \pi_1(S_g,p)$

(ϕ_* & Φ agree up to conjugacy)

[Will use the fact that γ_i form chain
↳ lifts of them to $\mathbb{H}^2$ are linked at ∞ $\forall i$
γ_i, γ_{i+1}]

in particular
(∵ γ_i, γ_{i+1} linked on surface S_g
⟹ take small nbhd of p ⟹ γ_i, γ_{i+1} linked on bndry of this nbhd.)

Denote $\underbrace{\underset{\text{auto}}{\phi_*^{-1}} \circ \Phi}_{\text{auto.}} =: F$ ⇝ WTS: $\exists \alpha \in \pi_1(S_g,p)$ s.t. $I_\alpha \circ F = id_{\pi_1(S_g)}$

⟹ F preserves linking at ∞.

$F(c_i) = c_i \ \forall i$

~~STATE~~

- $F(c_1) = c_1 \implies F(\gamma_1) = \alpha_1^{-1}\gamma_1\alpha_1$ for some $\alpha_1 \in \pi_1(S_g)$

conjugacy classes

$$\Rightarrow \boxed{I_{\alpha_1} \circ F(\gamma_1) = \gamma_1}$$

- $F(c_2) = c_2$

$I_{\alpha_1} \circ F$ pres linking at ∞ insert. ⟨Fact⟩ – pg. 8

γ_1, γ_2 are linked

$\implies I_{\alpha_1} \circ F(\gamma_2)$ linked at ∞ w/ γ_1

so $I_{\alpha_1} \circ F(\gamma_2) = \gamma_1^{-k}\gamma_2\gamma_1^{k}$ some $k \in \mathbb{Z}$

$$\implies I_{\gamma_1^k\alpha_1} \circ F(\gamma_1) = I_{\gamma_1^k} \circ I_{\alpha_1} \circ F(\gamma_1)$$

$$= I_{\gamma_1^k}(\gamma_1) = \gamma_1$$

$$I_{\gamma_1^k\alpha_1} \circ F(\gamma_2) = I_{\gamma_1^k} \circ I_{\alpha_1} \circ F(\gamma_2)$$

$$= I_{\gamma_1^k} \circ (\gamma_1^{-k}\gamma_2\gamma_1^{k}) = \gamma_2$$

⇝ Induction:

$$\underbrace{I_{\gamma_1^k\alpha_1}}_{\uparrow} \circ F(\gamma_i) = \gamma_i \quad \forall i \geqslant 3$$

desired inner auto.

$\Rightarrow \Phi_* \cong [\Phi]$.

└ surjectivity □.