Notre Dame Graduate Student Topology Seminar, Fall 2018

Mapping Class Groups and the Dehn-Nielsen-Baer Theorem: Problem Set

1. (Super easy warm-up) Show that the inner automorphism group $Inn(G)$ is a normal subgroup of $Aut(G)$.

2. (Tim's question) Recall that the map in the D-N-B theorem sends a homeomorphism $\phi : S \to S$, given any path $\gamma : p \rightsquigarrow \phi(p)$, to the composition

$$\pi_1(S,p) \xrightarrow{\phi_*} \pi_1(S,\phi(p)) \xrightarrow{\gamma_*} \pi_1(S,p).$$

As giving an element in $Out(\pi_1(S,p))$, this is independent of the choice of path γ, and is well-defined up to isotopy (i.e. defined on $Mod^{\pm}(S)$).

Show that this map is in fact a group homomorphism.

3. There were a couple bijections that I stated and caused confusion...I don't remember if it was both of these bijections, or if the confusion was totally cleared up, but here are the two that are used in the proof–clarify whichever (either, both, neither?) is still confusing:

$$\left\{ \begin{array}{l} \text{free homotopy classes of (unbased)} \\ \text{maps } S \to S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of homomor-} \\ \text{phisms } \pi_1(S) \to \pi_1(S) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{free homotopy classes of oriented} \\ \text{curves in } S_g \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of elements of} \\ \pi_1(S_g) \end{array} \right\}$$

[This last one corresponds to the picture of the unbased simply closed curves c_i and the generating elements of $pi_1(S_g,p), \gamma_i$ in the picture I drew...]

4. Why is it important to pass to the γ_i and not just use the c_i in the proof of surjectivity? (Hint: I think there should be some property of the γ_i, γ_{i+1} which causes them to be linked at infinity, whereas arbitrary lifts of the c_i, c_{i+1} are not necessarily linked at infinity...)

5. (Ambitious 'problem', if there's lots of extra time...) There is an alternative approach to the Dehn-Nielsen-Baer theorem inspired by 3-manifold theory. [See the Farb-Margalit book, section 8.3.1., available online through the ND library:

`https://www-jstor-org.proxy.library.nd.edu/stable/j.ctt7rkjw.13?refreqid=excelsior%3A6e8116926083a2b6fb488cdd66d9f8a6&seq=18#metadata_info_tab_contents`]

The set-up is as follows: Let S be a surface with $\chi(S) < 0$. Since S is a $K(\pi_1(S), 1)$-space, every outer automorphism of $\pi_1(S)$ is induced by some unbased map $S \to S$. By the Whitehead theorem and because $\pi_i(S) = 0, i > 1$, this map $S \to S$ is a homotopy equivalence. Thus the surjectivity of the D-N-B theorem reduces to the following:

Theorem 1. If $g \geq 2$, any homotopy equivalence $\phi : S_g \to S_g$ is homotopic to a homeomorphism.

Look at the Farb-Margalit proof of this, using pairs of pants decompositions of the surface S_g and the Alexander lemma.