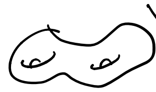
Lattices, mapping class groups, and ~~$\mathrm{Out}(F_n)$~~.

no time

Fundamental example in GGT:

$\Gamma = \pi_1\Big($  $\Big) \curvearrowright \mathbb{H}^2$ by isometries, properly, cocompactly.

geometry of $\mathbb{H}^2$ tells us about algebra of Γ

ex: word problem in Γ solvable, no $\mathbb{Z}\times\mathbb{Z}$ subgroups, etc...] spent a lot of time on this...

Note: $\mathbb{H}^2 \approx \mathrm{Isom}(\mathbb{H}^2)/\mathrm{Stab}(pt) \cong PSL_2\mathbb{R}/SO(2)$.

Generally: If G (conn., noncompact, no center) simple Lie group, $K \leq G$ maximal compact subgp, then G/K admits G-invt. Riemannian metric.

G/K is a symmetric space with curvature ≤ 0.

> On level of Lie algebras: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
>
> $\exists\, \sigma : \mathfrak{g} \to \mathfrak{g}$ s.t. $\sigma|_{\mathfrak{p}} = -\mathrm{Id}$, $\sigma|_{\mathfrak{k}} = \mathrm{Id}$.
>
> $\langle Y, Z\rangle := -\mathrm{Tr}\big(\mathrm{ad}(Y)\circ \mathrm{ad}(\sigma(Z))\big)$
>
> define $\mathfrak{k}$-invt. ~~positive definite quadratic form~~ inner product on $\mathfrak{g}$

Ex: $G = SO(1,n)^0 = \{A \in SL_{n+1}\mathbb{R} \mid A\, I_{1,n} A^T = I_{1,n}\}^0$ where $I_{1,n} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}$

Then $G/K \cong \mathbb{H}^n$, and $G = \mathrm{Isom}(\mathbb{H}^n)^0$

Ex: $G = SL_n\mathbb{R}$, $K = SO(n)$.

Then $X_n := G/K$ = {marked unit covolume lattices $\Lambda \leq \mathbb{R}^n$}/rotation

= [{positive definite quadratic forms on $\mathbb{R}^n$}] skip

= {marked flat unit volume n-tori}/isometry

Note: $SL_n\mathbb{Z} \backslash X_n$ is not compact (i.e. $SL_n\mathbb{Z}$ not cocompact)

So geometry of $SL_n\mathbb{Z}$ is not geometry of X_n.

Good news: $SL_n\mathbb{Z}$ is discrete, and $SL_n\mathbb{Z} \backslash X_n$ has finite volume. (reduction theory)

Call such $\Gamma \leq SL_n\mathbb{R}$ a lattice.

skipped

Fix: Add boundary to X_n to get $\overline{X_n}$ s.t. $SL_n\mathbb{Z} \curvearrowright \overline{X_n}$ cocompact.

(Raghunathan?) (Borel-Serre?) Souped-up version of adding S^1 as boundary of $\mathbb{H}^2$ as in proof of Mostow rigidity.

Corollary of Fix: $SL_n\mathbb{Z}$ has finite cohomological dimension.

Analyzing boundary data has many applications:

Thm: (Eskin, Farb, Schwartz) If Γ is a non-cocpt lattice in a simple Lie group G s.t. G/K has a 2-dim Euclidean plane, and Λ quasi-isometric to Γ, then Λ, Γ are commensurable up to finite kernels.

S_g = closed surface of genus g

<u>Another group</u>: $\mathrm{Mod}(S_g) := \mathrm{Homeo}^+(S_g)/_{\text{isotopy}}$

[Want to think of this like a lattice in a Lie group.]

What is analog of symmetric space?

<u>Teichmüller space</u>: Given S_g, $g \geq 2$, define

$$\mathcal{T}_g := \left\{ (X,f) \;\middle|\; f: S_g \xrightarrow{\approx} X,\ X \text{ is hyperbolic surface} \right\} \Big/ \sim$$

homeo

where $f_1 : S_g \to X_1 \sim f_2 : S_g \to X_2$ if $\exists$ isometry $I : X_1 \to X_2$

s.t.
$$\begin{tikzcd} & X_1 \arrow[d, "I"] \\ S_g \arrow[ur, "f_1"] \arrow[r, "f_2"'] & X_2 \end{tikzcd}$$
commutes up to htpy.

- See $\mathrm{Mod}(S_g) \curvearrowright \mathcal{T}_g$ by precomposition:

$$[\varphi]\cdot[(X,f)] := [(X, f\circ\varphi)].$$

<u>Note</u>: $\mathcal{T}_1 = \{\text{homeos } T^2 \to X \mid X \text{ flat torus of unit area}\}/\sim$

$= \{\text{marked lattices in } \mathbb{R}^2\}/_{\text{Euclidean isoms + homothethies}}$

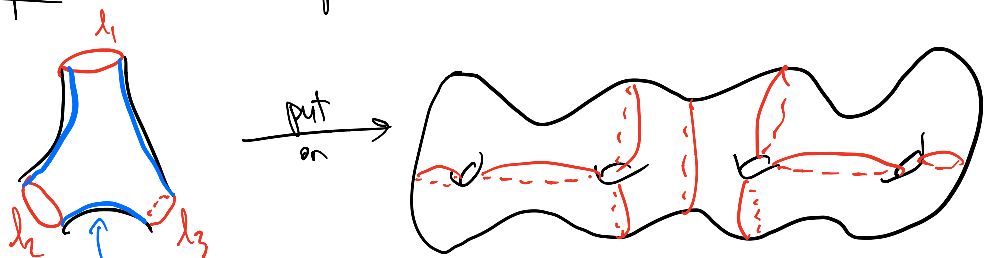
$= SL_2\mathbb{R}/SO(2)$

$= \mathbb{H}^2$

Theorem: If $g \geq 2$, $\mathcal{T}_g \approx \mathbb{R}^{6g-6}$. (Fenchel-Nielsen coords)

Idea of proof: Put on pants:

$P =$



seams minimize length between cusps

genus g surface wears $2(g-1)$ pairs of pants.

Hyperbolic metric on P $\longleftrightarrow$ waist/cuff lengths $\ell_1, \ell_2, \ell_3 \in \mathbb{R}^+$

This gives $\mathcal{T}_g \longrightarrow \mathbb{R}^{\frac{2(g-1)\cdot 3}{2}} = \mathbb{R}^{3g-3}$.

Other $3g-3$ dimensions come from twist as sew pants together.

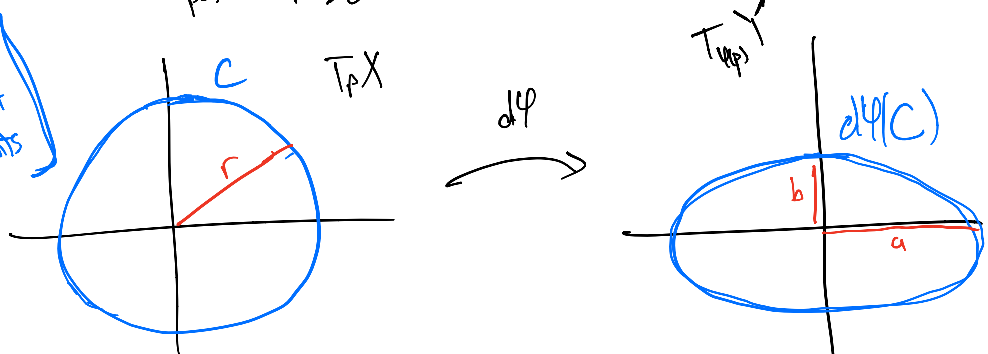
Teichmüller metric: Given $(X,f), (Y,h) \in \mathcal{T}_g$,

$h \circ f^{-1} : X \to Y$ is a homeom.

Thm: (Teichmüller, Ahlfors) $\exists!$ map $\varphi : X \to Y$ is homotopy class of $h \circ f^{-1}$ that has minimal dilatation $K(\varphi)$.

<u>Def'n</u>: $K(\varphi) = \sup\limits_{p\in X} \limsup\limits_{r\to 0} \left(\text{eccentricity of } d\varphi(\text{circle of radius } r)\right)$

can assume $d\varphi$ exists at all but fin. many points



eccentricity $= \dfrac{a}{b}$

$$\boxed{\underline{\text{Def'n}}:\ d_{\text{Teich}}(X,Y) := \log\left(K(\varphi)\right) \quad \text{for } \varphi \text{ as above.}}$$

<u>Facts</u>:

- $(\mathcal{T}_g, d_{\text{Teich}})$ is a <u>geodesic</u> metric space;
  given $X, Y \in \mathcal{T}_g$, $\exists\ \gamma : [0,1] \to \mathcal{T}_g$
  with $\gamma(0) = X$, $\gamma(1) = Y$, $d_T(\gamma(a), \gamma(b)) = |b-a|\ \forall a, b$.
- Induces topology $\mathcal{T}_g \approx \mathbb{R}^{6g-6}$
- Not a Riemannian metric, but has aspects of (large-scale) nonpositive curvature