GSTS Problem Session

Week 1

1. Show that the collection $\{T_g\}$ constructed last week is indeed a tiling.

2. Recall that the level m congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$ is defined as

$$\mathrm{SL}(2,\mathbb{Z})[m] := \ker(\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}/m\mathbb{Z})).$$

   The goal of this exercise is to show that $\mathrm{SL}(2,\mathbb{Z})[m]$ is free for $m \geq 3$. We will accomplish this by finding a tree on which $\mathrm{SL}(2,\mathbb{Z})[m]$ acts freely.

   The Farey Tree. We say that a pair $(m,n) \in \mathbb{Z}^2$ is *primitive* if $\gcd(|m|,|n|) = 1$. Define an equivalence relation $\sim$ on the set of primitive elements of $\mathbb{Z}^2$ by setting $(m,n) \sim -(m,n)$. The *Farey graph* is the the graph with vertex set

$$\{(m,n) \in \mathbb{Z}^2 : (m,n) \text{ is primitive}\}/\sim$$

   and two vertices $\pm(m,n)$ and $\pm(m',n')$ are connected by an edge if

$$\det\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = 1.$$

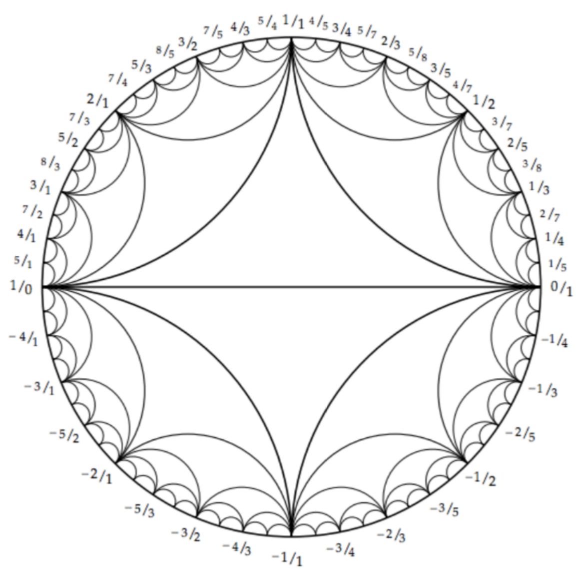

Figure 1: The Farey Graph

Form the *Farey Complex* by taking the Farey Graph and "filling in all the triangles". Finally, we define the *Farey Tree* to be the graph with vertex set consisting of triangles and edges of the Farey Complex and edges corresponding to inclusions of edges into triangles.

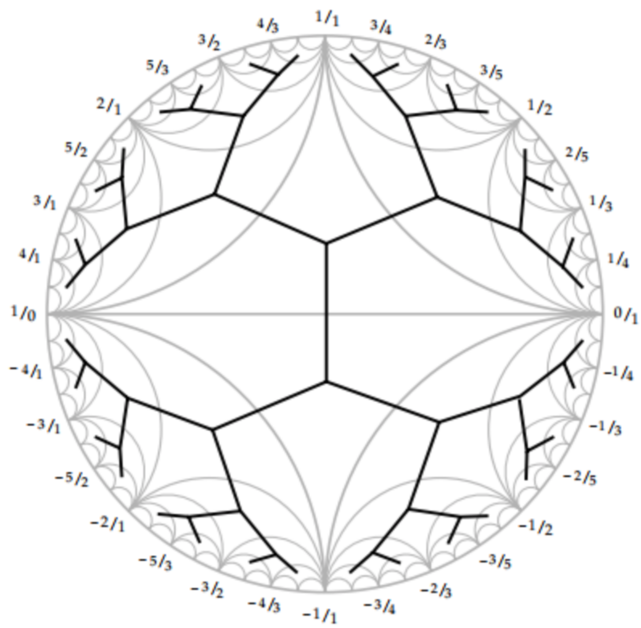

Figure 2: The Farey Tree

(a) Show that $\mathrm{SL}(2,\mathbb{Z})$ acts on the Farey tree.

(b) What are the vertex/edge stabilizers of this action?

(c) Conclude that $\mathrm{SL}(2,\mathbb{Z})[m]$ is free for $m \geq 3$.