# Geometric Group Theory

## Week 2

Last week, we saw that if a group $G$ acts freely on tree, then $G$ is free. This week, we will establish the converse of this theorem by introducing the Cayley graph of a finitely group.

**Definition 1.** Let $G$ be a finitely generated group with finite generating set $S$. Then Cayley graph of $G$ with respect to $S$, denoted $\Gamma(G, S)$, has one vertex for each element of $G$ and an edge from $g \in G$ to $h \in G$ is there is some $s \in S$ such that $gs = h$.

Notice that the Cayley graph gives a metric $d_S$ on the group $G$ called the *word metric*. More precisely, define $d_S(g, h)$ to be the distance in the Cayley graph $\Gamma(G, S)$ from $g$ to $h$ (where each edge has length 1.

**Example 1.** The Cayley graph of $F_2 = \langle a, b | \rangle$ is the 4-valent tree.

**Example 2.** The Cayley graph of $\mathbb{Z} = \langle 1 \rangle$ is the real line with vertices on the integer points.

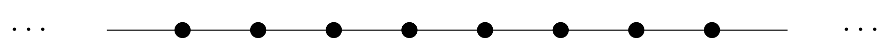
Since relations in a group correspond to loops in its Cayley graph, we have the following fact:

**Proposition 1.** The Cayley graph of a free group (with free generating set) is a tree.

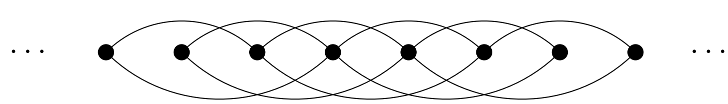
*Remark.* A free group acts freely and transitively on its Cayley graph. In other words, we have established the converse of the theorem proved last week.

So, we have constructed a graph on which finitely generated groups act. However, the Cayley graph depends on which generating set was chosen. It turns out that the Cayley graph can vary pretty wildly depending on which generating set we pick.

**Example 3.** Consider the generating sets $\{1\}$ and $\{2, 3\}$ for $\mathbb{Z}$. We saw that $\Gamma(\mathbb{Z}, \{1\})$ is the graph



but $\Gamma(\mathbb{Z}, \{2, 3\})$ is the graph



As we can see, the identity $(\mathbb{Z}, d_{\{1\}}) \to (\mathbb{Z}, d_{\{2,3\}})$ isn't even an isometry!

It would be convenient to talk about *the* Cayley graph associated to a group, but unfortunately two different Cayley graphs may not be isometric. Therefore, we would like to weaken our notion of equivalence to one in which Cayley graphs are unique. To do this, we note that, in our examples above, distances between vertices change, but not by much. So, we would like our notion of equivalence to capture this; namely, we want to allow for bounded stretches or shrinks. This sort of map is called a *bi-Lipshitz embedding*.

**Definition 2.** A map $f : X \to Y$ is a *bi-Lipshitz* embedding if there is some $K \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

If a bi-Lipshitz embedding is surjective, then it is called a *bi-Lipshitz equivalence*.

*Remark.* Notice that if $K = 1$, then $f$ is an isometry.

With this definition, we have:

**Theorem 1.** *If $G$ is a finitely generated group with finite generating sets $S$ and $S'$, then the identity map $(G, d_S) \to (G, d_{S'})$ is a bi-Lipshitz equivalence.*

*Proof.* Notice that $G$ acts by isometries on its Cayley graph, and so $d_S(g, h) = d_S(id, g^{-1}h)$. Therefore, it suffices to show that there is some $K \geq 1$ such that

$$\frac{1}{K} d_S(1, g) \leq d_{S'}(1, g) \leq K d_S(1, g)$$

for all $g \in G$. Since $S$ generates $G$, we can write $g = s_1 s_2 \cdots s_k$. This gives a path in the Cayley graph passing through the vertices

$$id, \quad s_1, \quad s_1 s_2, \quad \ldots, \quad s_1 s_2 \cdots s_k = g.$$

Notice that each of these vertices are adjacent in $\Gamma(G, S)$. We would like to get a bound on the distance between $s_1 s_2 \cdots s_i$ and $s_1 s_2 \cdots s_i s_{i+1}$ in $\Gamma(G, S')$. Let

$$K = \max\{d_{S'}(1, s) \mid s \in S \cup S^{-1}\},$$

which is well-defined since $S$ is finite. Then, applying the triangle inequality a bunch of times, we get

$$d_{S'}(1, g) \leq K d_S(1, g).$$

Switch $S'$ and $S$ to get the other inequality. □

So, we have shown that the metric space $(G, d_S)$ is bi-Lipshitz equivalent to $(G, d_{S'})$. However, this does not extend to the Cayley graphs $\Gamma(G, S)$ and $\Gamma(G, S')$. The first hurdle is even constructing a map $\Gamma(G, S) \to \Gamma(G, S')$. Where does it send points on the edges? The idea is going to be to "collapse" each edge by sending every point to the nearest vertex (will require some choices), and then "expand" out to $\Gamma(G, S')$. The problem is that this "collapsing" process is not bi-Lipshitz. To see why, let $f : \Gamma(G, S) \to \Gamma(G, S)$ be this collapsing map and consider two points $x$ and $y$ close to the center of an edge, but on opposite sides. Then we have $d_S(f(x), f(y)) \geq 1$. But, we can make $d_S(x, y)$ as small as we would like by choosing closer points, so $f$ is not bi-Lipshitz.

Therefore, we need to relax our notion of equivalence a bit more to allow for this small scale nastiness. This will be done by considering *quasi-isometries*.

**Definition 3.** A map $f : X \to Y$ is a quasi-isometric embedding if there is some $K \geq 1$ and $C \geq 0$ such that

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

for all $x_1, x_2 \in X$. Moreover, a quasi-isometric embedding $f$ is a quasi-isometry is there is some $D \geq 0$ such that for every $y \in Y$, there is some $x \in X$ such that $d_Y(f(x), y) \leq D$.

*Remark.* Note that $D = 0$ is equivalent to $f$ being surjective.

**Example 4.** The floor function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lfloor x \rfloor$ is a quasi isometry.

**Example 5.** The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x + 1000 & \text{if } x \in \mathbb{Z} \\ x & \text{else} \end{cases}$$

is a quasi-isometry.

Our goal has been to find an equivalence relation on metric spaces under which all Cayley graphs of a given group $G$ are equivalent. However, from the definitions, it is unclear that quasi-isometry gives an equivalence relation. This is taken care of by the following definition and exercise:

**Definition 4.** Given $f : X \to Y$, a quasi-isometry of $f$ is a map $g : Y \to X$ such that there exists some $k \geq 0$ so that

$$d_X(g(f(x)), x) \leq k \text{ for all } x \in X, \qquad \text{and} \qquad d_Y(f(g(y)), y) \leq k \text{ for all } y \in Y.$$

**Exercise 1.** Show that $f$ is a quasi-isometry if and only if $f$ has a quasi-inverse $g$. Moreover, show that the quasi-inverse of a quasi-isometry is also an quasi-isometry.

With these definitions, we have:

**Theorem 2.** *If $G$ is a finitely-generated group with finite generating sets $S$ and $S'$, then $\Gamma(G,S)$ is quasi-isometric to $\Gamma(G,S')$.*

*Proof.* Just as we stated before, consider the composition

$$\Gamma(G,S) \xrightarrow{f} (G,d_S) \xrightarrow{id} (G,d_{S'}) \xrightarrow{g} \Gamma(G,S'),$$

where $f$ is this collapse map discussed before, and $g$ is the quasi inverse of the collapse map $\Gamma(G,S') \to (G,d_{S'})$. Note that $id : (G,d_S) \to (G,d_{S'})$ is an quasi-isometry since it is a bi-Lipshitz equivalence. □

**Exercise 2.** Show that the composition of quasi-isometries is a quasi-isometry.

*Remark.* With this theorem, it makes sense to talk about groups being quasi-isometric to metric spaces (or other groups). More concretely, we say a group $G$ is quasi-isometric to a metric space $X$ if any Cayley graph of $G$ is quasi-isometric to $X$.

Great! So, we have found that the Cayley graph of a graph is unique up to quasi-isometry. But how does that tie in to the algebraic properties of groups (recall the goal is geometric group theory is to study algebraic properties via actions on groups). One interesting fact connecting algebraic properties of groups with quasi-isometries is called the Milnor-Schwarz Lemma. Before we state the lemma, we will recall a few definitions from metric spaces and group actions.

**Definition 5.** (From metric spaces)

- A metric space $(X,d)$ is a *geodesic metric space* if for any two points $x_1, x_2 \in X$, there is a geodesic segment $\gamma : [a,b] \to X$ such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$.
- A metric space is *proper* if for all $x \in X$, there is some $r > 0$ such that the closed ball $\overline{B(x,r)} \subseteq X$ is compact.

**Definition 6.** (From group actions)

- An action of $G$ on $X$ is *properly discontinuous* if for all compact subsets $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ if finite.
- An action of $G$ on $X$ is *cocompact* if for all $x \in X$, there is some $R > 0$ such that $G \cdot \overline{B(x,R)} = X$.
- An action of $G$ on $X$ is *geometric* if it is properly discontinuous, cocompact, and $G$ acts by isometries.

**Theorem 3.** *(Milnor-Schwarz Lemma) Let $G$ be a group and $(X,d)$ a proper geodesic metric space. If $G$ acts on $X$ geometrically, then $G$ is finitely-generated and $G$ is quasi-isometric to $X$.*

**Definition 7.** Two groups $G$ and $G'$ *differ by finite groups* if either

- $G$ is isomorphic to a finite index subgroup of $G'$, or
- $G$ is isomorphic to a quotient of $G'$ by a finite group.

**Corollary 1.** If two groups $G$ and $G'$ differ by finite groups, then $G$ and $G'$ are quasi-isometric.

The converse of the corollary is in general false. However, there are certain classes of groups for which the converse is true.

**Definition 8.** A group $G$ is quasi-isometrically rigid if every group quasi-isometric to $G$ differs from $G$ by a finite groups.

To finish, we remark that $\mathbb{Z}^n$ is quasi-isometrically rigid. In particular,

**Theorem 4.** *If $G$ is quasi-isometric to $\mathbb{Z}^n$, then $G$ has a finite-index subgroup isomorphic to $\mathbb{Z}^n$.*