GSTS Problem Session

Week 2

1. Recall that a *quasi-inverse* of a function $f : X \to Y$ is a function $g : Y \to X$ such that there is some $k \geq 0$ such that $d_X(g(f(x))) \leq k$ for all $x \in X$ and $d_Y(f(g(y))) \leq k$ for all $y \in Y$.
   (a) Show that a map $f : X \to Y$ is a quasi-isometry if and only if f has a quasi-inverse.
   (b) Show that a quasi-inverse of a quasi-isometry is also a quasi-isometry.
2. Let G be a finitely generated group with finite generating set S. For any subset A of vertices of $\Gamma(G, S)$, define the *S-boundary* of A by

$$\partial_S A = \{g \in G \mid g \notin A \text{ and } d_S(g, a) = 1 \text{ for some } a \in A\}.$$

   We say that G is *S-amenable* if for any $\varepsilon > 0$, there is some finite subset A of vertices of $\Gamma(G, S)$ such that $|\partial_S A| < \varepsilon |A|$.
   (a) Show that $\mathbb{Z}$ is both $\{1\}$-amenable and $\{2, 3\}$-amenable.
   (b) Show that any finite group G is S-amenable for any finite generating set S.
   (c) Suppose G_1 and G_2 are finitely generated groups with finite generating sets S_1 and S_2. Show that if G_1 is S_1-amenable and G_2 is S_2-amenable, then $G_1 \times G_2$ is S-amenable, where

$$S = \{(s_1, id_{G_2}) \mid s_1 \in S_1\} \cup \{(id_{G_1}, s_2) \mid s_2 \in S_2\}.$$

   (d) Let G be a finitely generated group with finite generating sets S and S'. Show that G is S-amenable if and only if G is S'-amenable. Thus, it makes sense to say a finitely generated group is *amenable* if it is S amenable for any finite generating set S.
   (e) Show that all finitely generated abelian groups are amenable.
   (f) Show that F_2 is not amenable.
   (g) Suppose that G and H are quasi-isometric finitely generated groups. Show that G is amenable if and only if H is amenable.

Note: These exercises were taken from Office Hour Seven of Office Hours with a Geometric Group Theorist by Clay and Margalit.