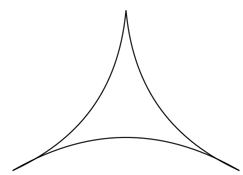
Geometric Group Theory

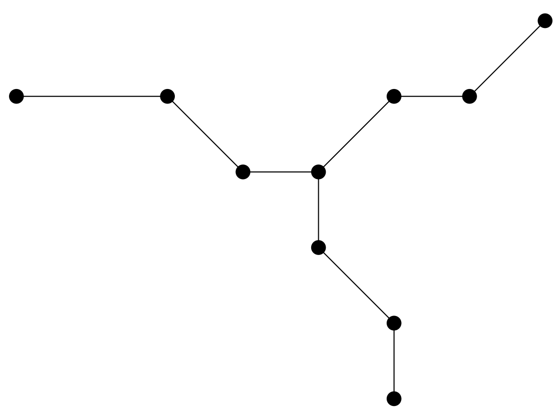
Week 3

Hyperbolic Groups

The topic of this lecture is going to be *hyperbolic groups*. When you first read "hyperbolic groups", you probably first thought of hyperbolic space. The two are, perhaps not so surprisingly, related. We first point out a key feature of hyperbolic space: geodesic triangles in hyperbolic space are "thin".



Before we delve further into what "thin" means for triangles in hyperbolic space, let's take a look at an extreme example – trees. As seen below, geodesic triangles in trees are (perhaps degenerate) tripods.



Consider a geodesic triangle in a tree with sides α, β, γ. Notice that $\alpha \cup \beta \supseteq \gamma$, and similarly for the other arrangements of α, β, γ. This is going to be our motivation for the "thinness" of triangles in a metric space. Of course, we don't want to require that one side of a geodesic triangle be contained in the union of the other two, otherwise geodesic triangles in hyperbolic space would not be thin. We would however, like this to be *almost* true.

Definition 1. A geodesic triangle with sides α, β, γ in a metric space is δ-*thin* if $N_\delta(\alpha \cup \beta) \supseteq \gamma$, $N_\delta(\alpha \cup \gamma) \supseteq \beta$, and $N_\delta(\beta \cup \gamma) \supseteq \alpha$, where $N_\delta(U)$ denotes the closed δ-neighborhood of U.

Now that we have defined what it means for triangles to be thin, we can define what it means for spaces to be hyperbolic.

Definition 2. A (geodesic) metric space is δ-hyperbolic if all geodesic triangles are δ-thin.

Example 1. (a) Trees are 0-hyperbolic.

(b) Hyperbolic space $\mathbb{H}^2$ is δ-hyperbolic for some $\delta \geq 0$ (what is the best δ?).

(c) $\mathbb{R}^2$ is not δ-thin for any $\delta \geq 0$.

Note that our definition of δ-hyperbolicity relies on the metric space being geodesic. However, there are several other equivalent definitions of δ-hyperbolic (Gromov's four-point condition) that only uses the metric, not geodesic triangles. There are many other equivalent definitions as well (for instance, the insize definitions). The problem with comparing these definition's is that the δ is not consistent throughout all definitions. It is true, however, that if a metric space X is δ-hyperbolic for one definition, then for any of the other definitions, X will be δ'-hyperbolic for some $\delta' \geq 0$. To compensate for this disparity, we make the following definition.

Definition 3. A metric space is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

Now, a space being hyperbolic does not depend on which definition one uses for δ-hyperbolic. Moreover, hyperbolicity is also preserved under quasi-isometry.

Theorem 1. *Suppose X and Y are quasi-isometric metric spaces. If X is hyperbolic, then Y is also hyperbolic.*

We won't be proving this theorem, however it should seem reasonable: if X is δ-hyperbolic and Y is quasi-isometric to X, then we should be able to find some δ' depending on δ and the error terms K and C from the quasi-isometry such that Y is δ'-hyperbolic.

Note that, with the theorem, the following definition is unambiguous.

Definition 4. A group is hyperbolic if any of its Cayley graphs are.

Example 2. (a) Free groups are hyperbolic.

(b) Finite groups are hyperbolic.

(c) $\mathbb{Z}^2$ is not hyperbolic.

Further Examples of Hyperbolic Groups

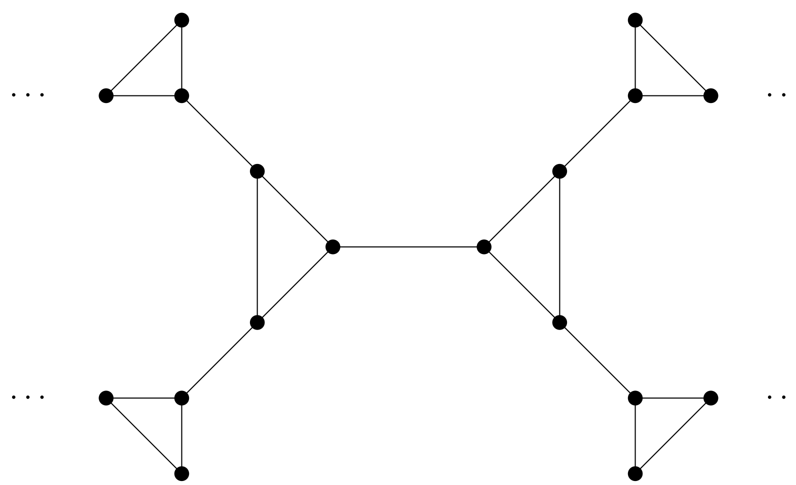
Special Linear Group

The examples we have seen so far have been somewhat trivial. So, let's see some more interesting examples. The first we will discuss is $\mathrm{PSL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$.

Recall that $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$. Notice that $\mathrm{PSL}(2,\mathbb{Z})$ also acts on the Farey Tree (discussed in the first problem session). However, the vertex stabilizers will no longer be trivial. They will, however, be finite. Using a similar technique to the one we used to prove groups acting freely on trees are free, one can show that groups acting on trees with finite vertex stabilizers are isomorphic to a free product. That idea can be applied here to show that

$$\mathrm{PSL}(2,\mathbb{Z}) \cong \langle a, b \mid a^2, b^3 \rangle \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z},$$

where $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. So, the Caley graph of $\mathrm{PSL}(2,\mathbb{Z})$ with respect to the generating set $\{a, b\}$ is:



Notice that this looks like a tree, except with all the vertices replaced by triangles. One can show that this graph is 1-hyperbolic (check by drawing some triangles). Thus, $\mathrm{PSL}(2,\mathbb{Z})$ is hyperbolic! Moreover, $\mathrm{PSL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ differ by finite groups ($\mathrm{PSL}(2,\mathbb{Z})$ is the quotient of $\mathrm{SL}(2,\mathbb{Z})$ by a finite group). Therefore, by Milnor-Schwarz, $\mathrm{PSL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{Z})$ are quasi-isomorphic, and thus $\mathrm{SL}(2,\mathbb{Z})$ is also hyperbolic.

Surface Groups

The final example we will discuss are surface groups (fundamental groups of closed surfaces). Recall that, for the closed surface of genus g, denoted Σ_g, we have

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2]\cdots[a_g, b_g] \rangle.$$

For $g = 1$, we have $\pi_1(\Sigma_1) = \pi_1(T^2) = \mathbb{Z}^2$, which we already have seen is not hyperbolic. However, we claim that the rest are.

Theorem 2. *The surface groups $\pi_1(\Sigma_g)$ are hyperbolic for $g \geq 2$.*

Proof. The method of proof will rely on covering spaces. We will discuss the proof in the $g = 1$ case first, and then translate to $g \geq 2$.

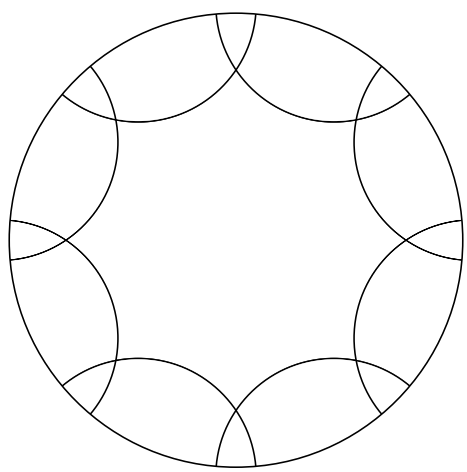
Recall that the universal cover of $\Sigma_1 = T^2$ is $\mathbb{R}^2$. In other words, $\Sigma_1 = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by integer translations vertically and horizontally. A fundamental domain for this action is the square S with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. We see that this action is:

- by isometries,
- cocompact, since $\mathbb{R}^2/\mathbb{Z}^2 = \Sigma_1$ is compact, and
- properly discontinuous, since only 8 translations of S intersect S.

Therefore, by Milnor-Schwarz, $\pi_1(\Sigma_1) = \mathbb{Z}^2$ is quasi-isometric to $\mathbb{R}^2$, which again shows that $\mathbb{Z}^2$ is not quasi-isometric.

In the $g \geq 2$ case, we now have that the universal cover is $\mathbb{H}^2$. Using the process above, we will show that $\pi_1(\Sigma_g)$ is quasi-isometric to $\mathbb{H}^2$. We will focus on the $g = 2$ case, but the other cases are similar.

For $g = 2$, we have that $\Sigma_2 = \mathbb{H}^2/\pi_1(\Sigma_2)$. A fundamental domain for this action is an octagon:



It is clear that this action is still by isometries, cocompact, and properly discontinuous. Therefore, by Milnor-Schwarz, $\pi_1(\Sigma_2)$ is quasi-isometric to $\mathbb{H}^2$. The same argument works for $g > 2$, except instead of an octagon in the universal cover, you will have a $(4g)$-gon. $\square$

The Word Problem

We will finish by discussing an actual algebraic property of hyperbolic groups – namely, that they have a solvable word problem. Recall the word problem for a group G: given a presentation for G and a word w in the generators of this presentation, is there a finite time algorithm to determine if w represents the trivial element of G?

A common method, introduced by Dehn, to show that groups have solvable word problem is to show that they admit a special type of presentation, called a *Dehn presentation.*

Definition 5. A finite presentation $G = \langle a_1, a_2, \ldots, a_n \mid r_1, r_2, \ldots, r_k \rangle$ is a *Dehn presentation* if the following conditions hold:

(a) There exist string $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$ in the a_j such that $r_i = u_i v_i^{-1}$ for all i (this just says that u_i and v_i represent the same element of G).

(b) The word length of v_i is strictly less than the word length of u_i.

(c) For any nontrivial reduced word w representing the identity in G, u_i or u_i^{-1} appears as a substring for some i.

We note that the existence of such a presentation solves the word problem as follows: given some nontrivial word w in the a_i,

- By (c), if w does not contain any u_i or u_i^{-1} then w does not represent the identity element of G.
- If w does contain some u_i or u_i^{-1}, replace this string with v_i or v_i^{-1}. By (a), this does not change the element of G represented by w. By (b), this makes w shorter.
- Repeat until w no longer contains any u_i or u_i^{-1} as a substring or you get to the trivial word.

It may seem like these conditions (especially (c)) are very restrictive. It requires that *every* string not representing the identity contains at least one of a finite list of substring. Amazingly, one can show that hyperbolic groups do have Dehn presentations.

Theorem 3. *Hyperbolic groups admit Dehn presentations.*

Before we prove the theorem, we will require a techincal definition and lemma.

Definition 6. A path γ in a graph is an m-local geodesic if every subpath of length m is a geodesic.

For example the the path around the 8-circuit is a 4-local geodesic. These should be though of as "almost geodesics".

Lemma. There are no (8δ)-local geodesic loops of length greater then or equal to 8δ in a δ-hyperbolic space.

We will not prove this lemma, but we can give an intuitive argument. In a hyperbolic space, geodesics like to diverge out to infinity (for contrast, recall that geodesics in positively curved manifolds such as the sphere tend to loop around on themselves). Therefore, in a hyperbolic space, a loop cannot be very close to being a geodesic, and this lemma quantifies just how far away from being a geodesic a loop must be.

Proof of theorem. Let G be a hyperbolic group with finite generating set, and suppose the Cayley graph of G with respect to S is δ-hyperbolic. Fix some integer $K > 8\delta$. Let $R = \{u_i v_i^{-1}\}$, where

- u_i ranges over all nonminimal words in $S \cup S^{-1}$ of length less than or equal to K, and
- v_i is a word of minimal length representing u_i.

We first show that $\langle S \mid R\rangle$ is a presentation for G. It is clear by construction that all elements of R are actual relations in G, so we only need to check that R includes all the necessary relations. By the way we've constructed R, it will contain all relations of length less than or equal to K. There is a relation r of length greater than K. Then, by the lemma, the loop in the Cayley graph represented by r cannot be an (8δ)-local geodesic. In particular, it contains a subpath of length at most 8δ which is not a geodesic. This path is represented by one of the u_i. Replace this u_i with the corresponding v_i to get a strictly shorter loop (note, this corresponds to inserting the relation $u_i^{-1} v_i$ into r). Repeat this process until you get a relation of length at most K. By construction, this relation will be in R, and we can recover r from it by inserting the corresponding $u_i v_i^{-1} \in R$. Thus, r is redundant with R, and so $G = \langle S \mid R\rangle$.

Now, we must check that $G = \langle S \mid R\rangle$ is a Dehn presentation. Condition (a) and (b) are satisfied by the way we constructed R. For (c), suppose that w is a nontrivial word in $S \cup S^{-1}$ and which represents the identity in G. If w has word length at most K, then $w \in R$ by construction, and so w contains some u_i as a substring. If w has word length greater than K, then the loop in the Cayley graph representing this word cannot be an (8δ)-local geodesic. In particular, it contains a subpath of length at most 8δ which is not a geodesic. This subpath corresponds exactly with one of our u_i. □