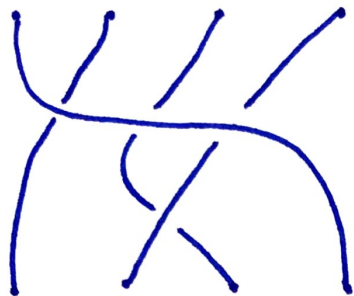
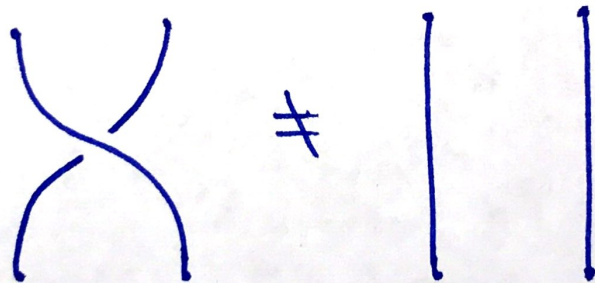
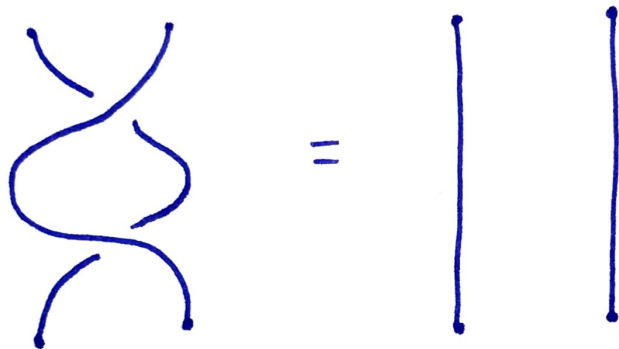


# Braid Groups

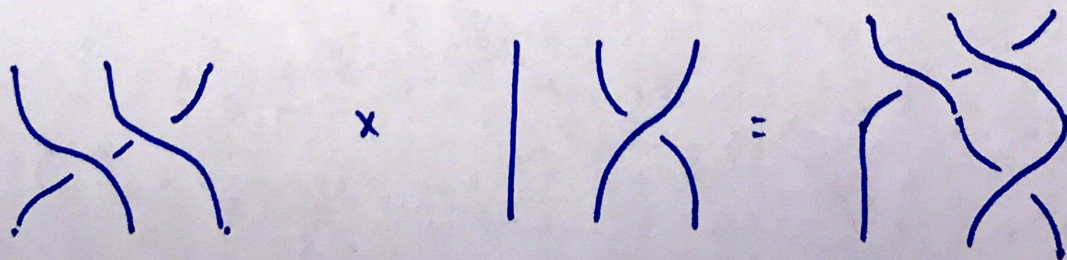
Braid:



Consider the set  $B_n = \{\text{braids on } n \text{ strings}\} / \text{isotopy rel endpoints}$



$B_n$  is a group under stacking



What are inverses & identity?

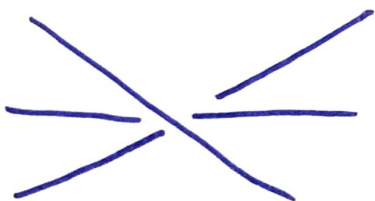
$$\sigma_i \times \sigma_i^{-1} = \text{three parallel vertical lines}$$

Flip upside-down!

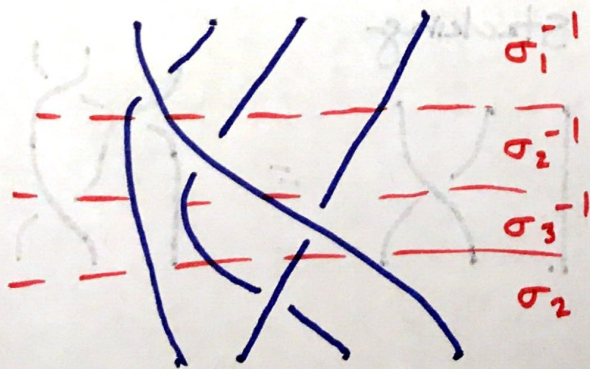
Building blocks:

$$\sigma_i = \text{strand } i-1 \text{ and } i+1 \text{ parallel, strands } i \text{ and } i+1 \text{ cross, strands } i+2 \text{ and } n \text{ parallel}$$

Remark: Any braid can be perturbed to avoid



- Can also perturb so that only one crossing happens at a given height



Prop: The  $\sigma_i$  generate  $B_n$ .

Relations: • If  $|i-j| > 1$ , then  $\sigma_i \sigma_j = \sigma_j \sigma_i$

$$\sigma_i \sigma_j = \begin{array}{c} \text{Diagram 1: Two vertical lines, the left one crosses over the right one.} \end{array} = \begin{array}{c} \text{Diagram 2: Two vertical lines, the right one crosses over the left one.} \end{array} = \sigma_j \sigma_i$$

• How about  $\sigma_i \sigma_{i+1} \neq \sigma_{i+1} \sigma_i$

$$\begin{array}{c} \text{Diagram 3: Left line crosses over right line.} \end{array} \neq \begin{array}{c} \text{Diagram 4: Right line crosses over left line.} \end{array}$$

• Braid Relation:  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

$$\begin{array}{c} \text{Diagram 5: Left line crosses over right line, then right line crosses over left line.} \end{array} = \begin{array}{c} \text{Diagram 6: Right line crosses over left line, then left line crosses over right line.} \end{array}$$

Theorem:  $B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i=1, \dots, n-2 \rangle$

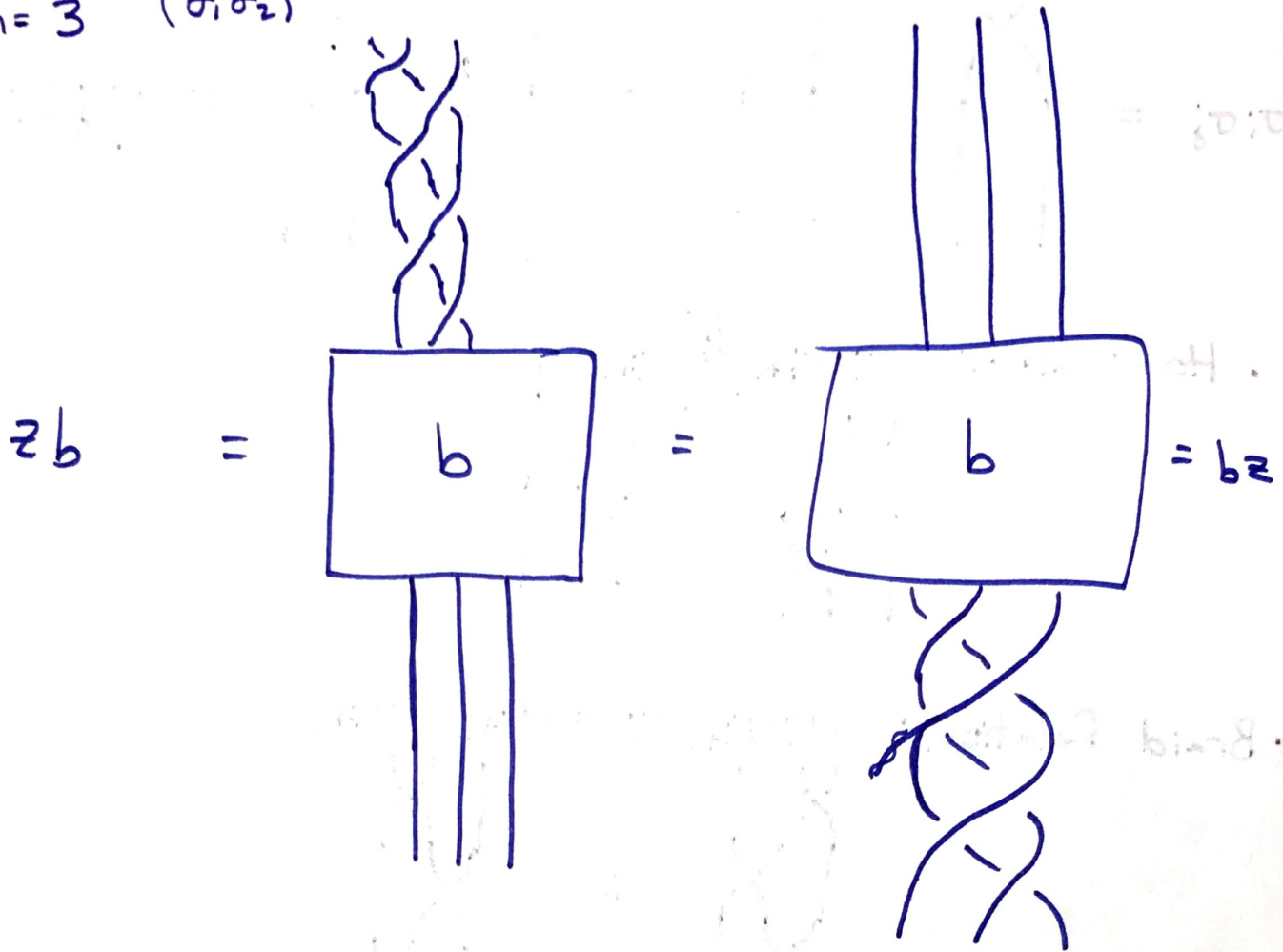
Ex: •  $B_1 = \{1\}$

•  $B_2 = \langle \sigma_1 \mid \sigma_1^2 = 1 \rangle \cong \mathbb{Z}_2$

•  $H_1(B_n) \cong \mathbb{Z}$

Center: Let  $z = (\sigma_1 \dots \sigma_{n-1})^n$

$n = 3 \quad (\sigma_1 \sigma_2)^3$



Prop:  $Z(B_n) = \langle z \rangle$  for  $n \geq 3$ .

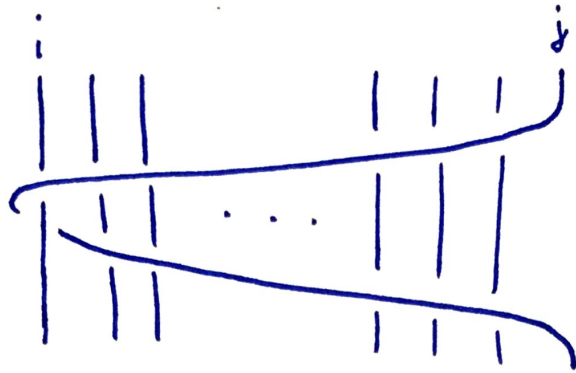
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Pure Braids: Have homomorphism  $\pi: B_n \rightarrow S_n$

$\ker \pi =: PB_n$  (pure braid group)

What is a genset for  $PB_n$ ?

Let  $a_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \cdots \sigma_{i+1})^{-1}$  for  $1 \leq i < j \leq n$



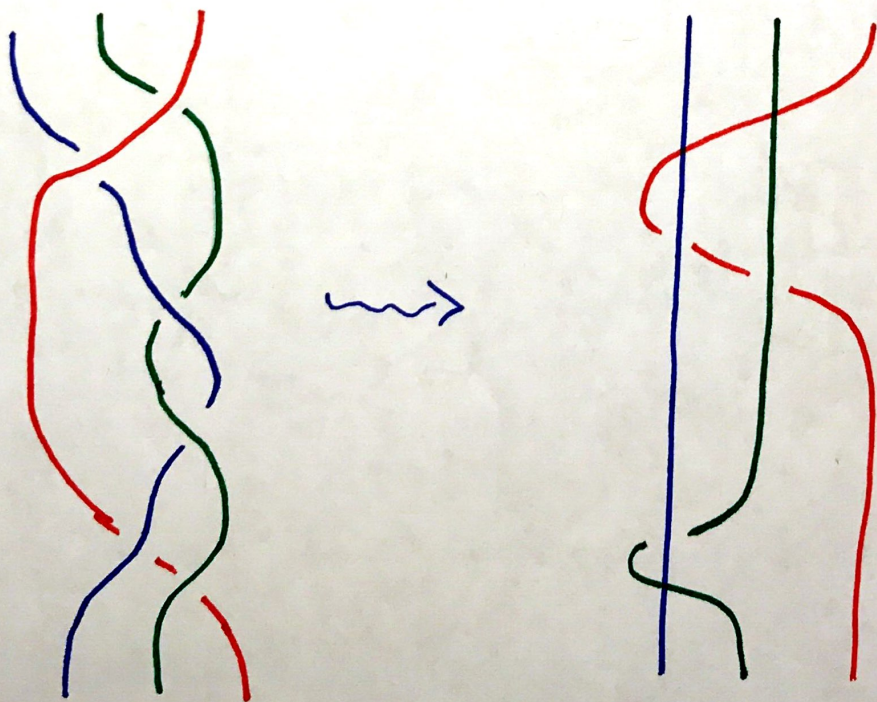
Prop: The  $a_{i,j}$  generate  $PB_n$

Notice:  $z \in PB_n$

$$z = (a_{1,2} a_{1,3} \cdots a_{1,n}) (a_{2,3} a_{2,4} \cdots a_{2,n}) \cdots (a_{n-2,n-1} a_{n-2,n}) a_{n-1,n}$$

How to tell if a pure braid is trivial:

Combing:



Claim: If two braids are equal, then they have the same combing

In particular, a <sup>pure</sup> braid is trivial if and only if its combing is the trivial braid (in the form  $||| \dots |$ )

Remark: Thus, this gives a solution to the word problem for braid groups