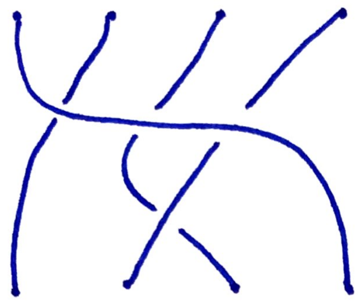
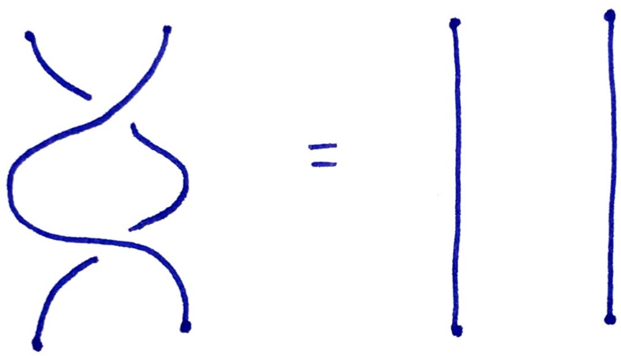
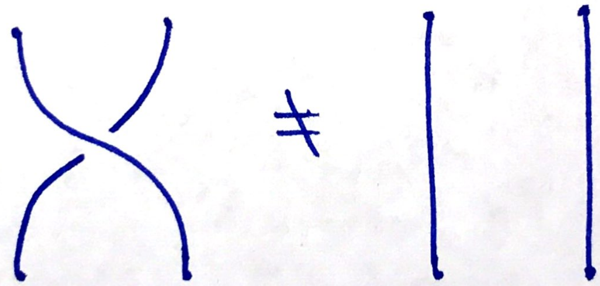
Braid Groups

Braid:

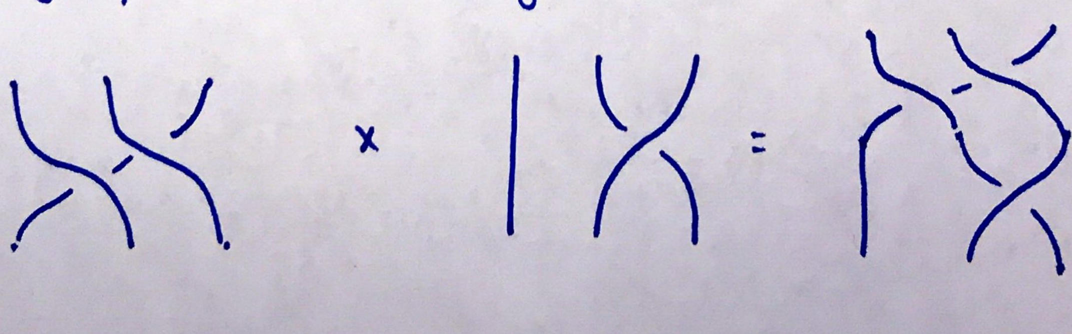


Consider the set $B_n = \{\text{braids on } n \text{ strings}\}/_{\text{isotopy rel endpoints}}$

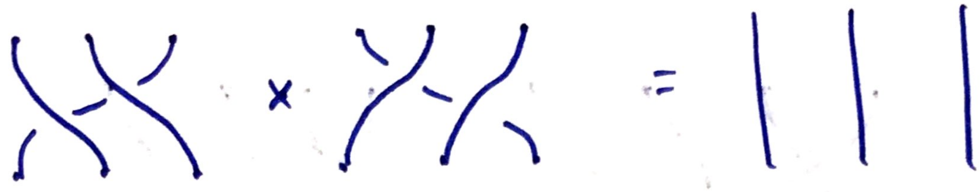




B_n is a group under stacking

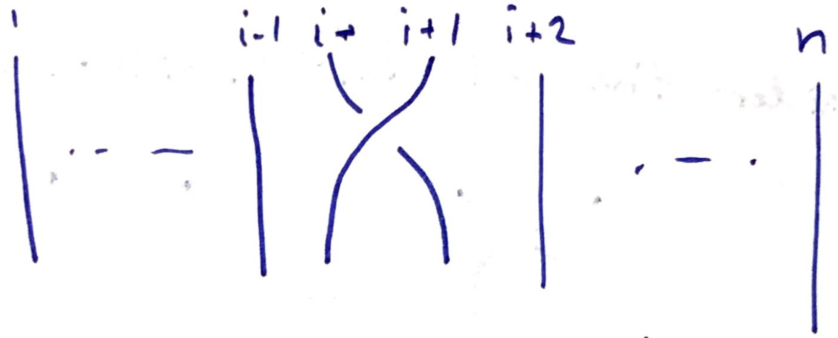


What are inverses / identity?

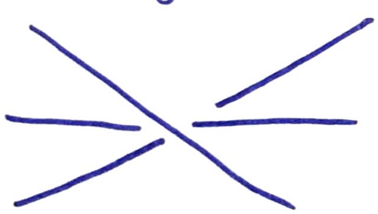


Flip upside-down!

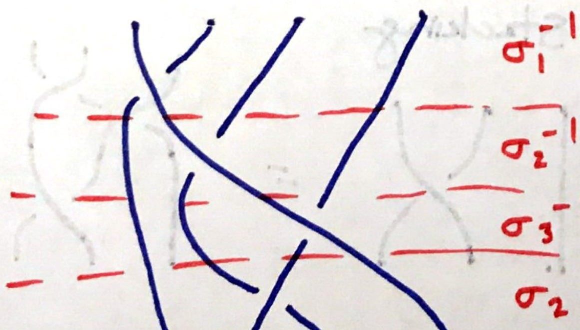
Building blocks:

$\sigma_i =$ 

Remark: · Any braid can be perturbed to avoid

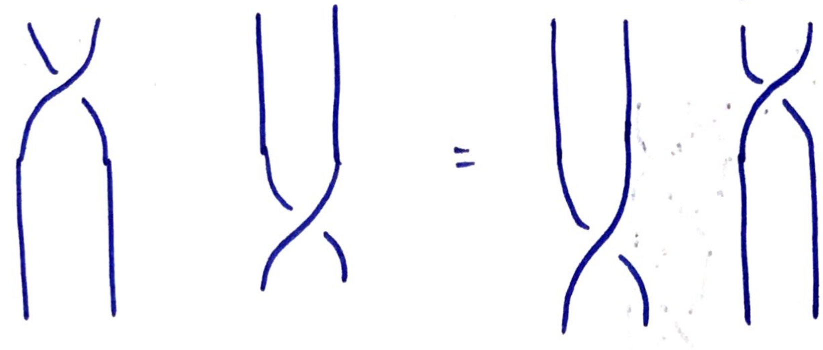


· Can also perturb so that only one crossing happens at a given height



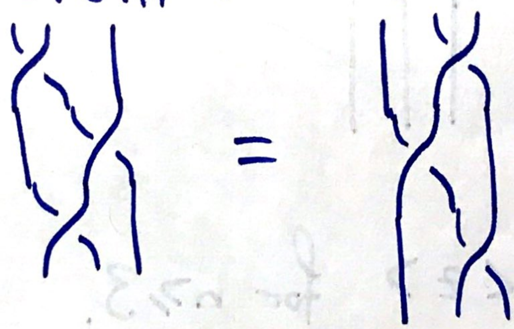
Prop: The σ_i generate B_n.

Relations: • If $|i-j|>1$, then $\sigma_i\sigma_j = \sigma_j\sigma_i$

$\sigma_i\sigma_j =$  $= \sigma_j\sigma_i$

• How about $\sigma_i\sigma_{i+1} \cancel{=} \sigma_{i+1}\sigma_i$

• Braid Relation: $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$



<u>Theorem</u>: $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1, \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i=1,\ldots,n-2 \rangle$

Ex: • $B_1 = \{1\}$

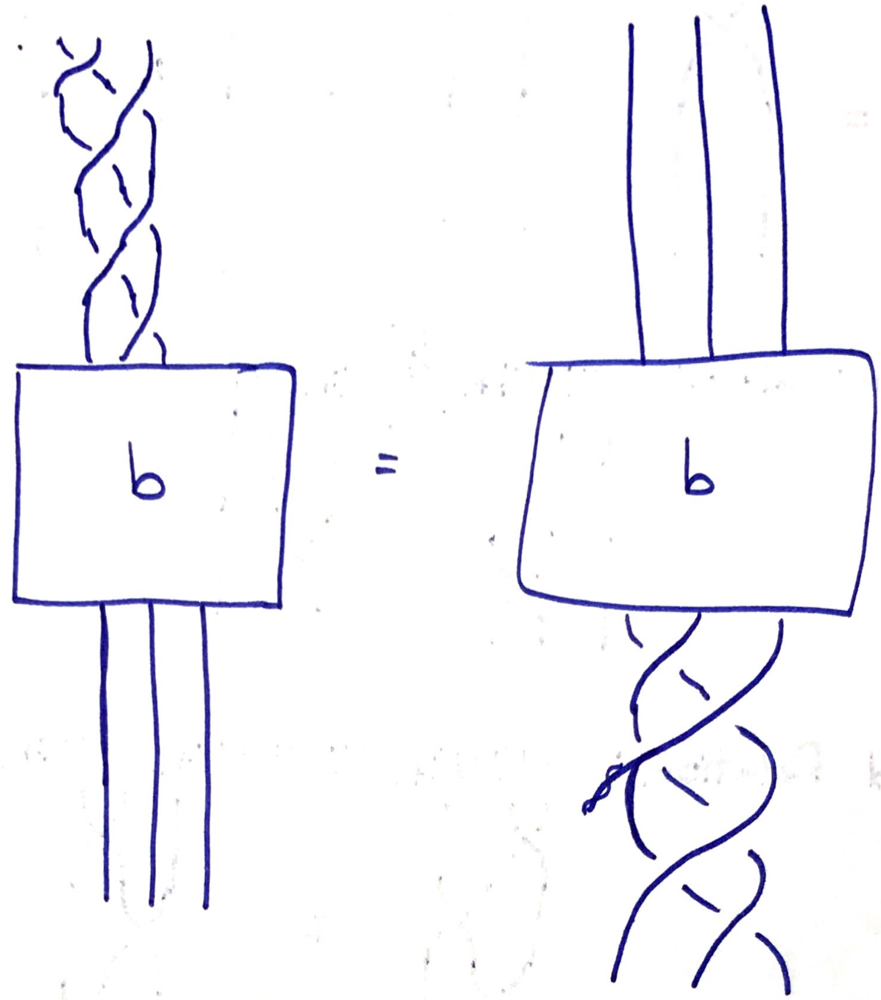
• $B_2 = \cancel{\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle} = \langle \sigma_1 \rangle \cong \mathbb{Z}$.

• $H_1(B_n) \cong \mathbb{Z}$.

Center: Let $z = (\sigma_1 \cdots \sigma_{n-1})^n$

$n = 3 \quad (\sigma_1\sigma_2)^3$

$zb =$



$= bz$

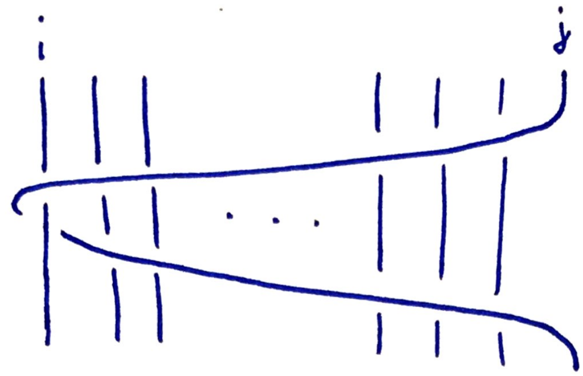
Prop: $Z(B_n) = \langle z \rangle$ for $n \geq 3$.

Pure Braids: Have homomorphism $\pi : B_n \to S_n$

$\ker \pi =: PB_n$ (pure braid group)

What is a genset for PB_n?

Let $a_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \cdots \sigma_{i+1})^{-1}$ for $1 \leq i < j \leq n$



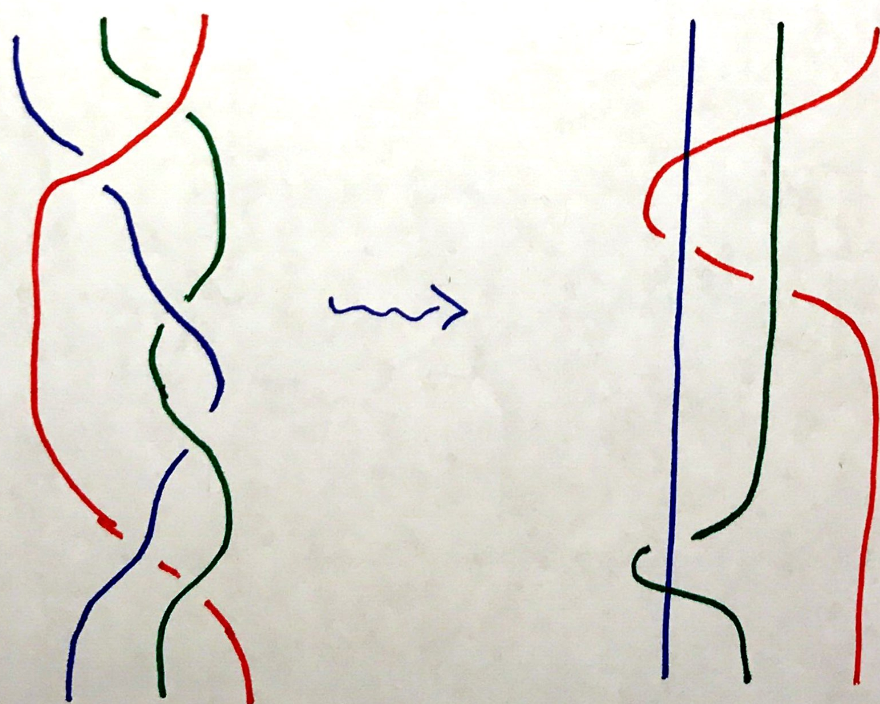
Prop: The $a_{i,j}$ generate PB_n

Notice: $z \in PB_n$

$$z = (a_{1,2}\, a_{1,3} \cdots a_{1,n})(a_{2,3}\, a_{2,4} \cdots a_{2,n}) \cdots (a_{n-2,n-1}\, a_{n-2,n})\, a_{n-1,n}$$

How to tell if a pure braid is trivial:

Combing:



Claim: If two braids are equal, then they have the same combing

In particular, a pure braid is trivial if and only if its combing is the trivial braid (in the form $|||\cdots|$)

Remark: Thus, this gives a solution to the word problem for braid groups